October 28 Math 2306 sec. 51 Fall 2024

Section 11: Linear Mechanical Equations

The basic model for the position, $x(t)$, at time t of an object of mass m suspended from a flexible spring, possibly subjected to linear damping or external forcing is

$$mx'' + bx' + kx = f(t),$$

where b is the damping coefficient, k the spring constant, and f an external driving force.

In the absence of damping, $b = 0$. In the absence of driving, $f(t) = 0$.

Forced Undamped Motion and Resonance

If damping is negligible, and the system is driven by a simple oscillator, $f(t) = f_0 \cos(\gamma t)$ or $f(t) = f_0 \sin(\gamma t)$, the differential equation becomes

$$x'' + \omega^2 x = F_0 \cos(\gamma t) \quad \text{or} \quad x'' + \omega^2 x = F_0 \sin(\gamma t),$$

where $\omega^2 = \frac{k}{m}$ and $F_0 = \frac{f_0}{m}$.

Pure resonance occurs when the forcing frequency γ matches the natural frequency ω.

$$\gamma = \omega$$

Even near resonance, $\gamma \approx \omega$, gives rise to large amplitude oscillations and can cause damaging vibrations.

Example

A 3 kg mass is attached to a spring with spring constant 48 N/m. Assume damping is negligible and the system is driven by a force $f(t) = F_0 \cos(\gamma t)$.

(a) What value of γ would induce pure resonance?

We found that $\omega^2 = \dfrac{k}{m} = \dfrac{48}{3} = 16$. So the resonance frequency $\gamma = 4$.

$x'(0)=0$ $x(0)=0$

(b) If the object starts from rest from the equilibrium position, find the displacement in the pure resonance case.

The differential equation in the resonance case is

$$3x'' + 48x = F_0 \cos(\omega t) \quad \Longrightarrow \quad x'' + 16x = \frac{F_0}{3}\cos(4t).$$

The initial conditions are $x(0) = x'(0) = 0$.

$$x'' + 16x = \frac{F_0}{3}\cos(4t),\ x(0) = 0,\ x'(0) = 0$$

The complementary solution is

$r^2+16=0$

$r=\pm 4i$

$$x_c = c_1\cos(4t) + c_2\sin(4t).$$

We were finding x_p in the form

$$x_p = At\cos(4t) + Bt\sin(4t).$$

Note that

$$\begin{aligned}
16 \quad x_p &= At\cos(4t) + Bt\sin(4t)\\
x_p' &= A\cos(4t) + B\sin(4t) - 4At\sin(4t) + 4Bt\cos(4t)\\
1 \quad x_p'' &= -8A\sin(4t) + 8B\cos(4t) - 16At\cos(4t) - 16Bt\sin(4t)
\end{aligned}$$

collect like terms as we substitute

$$x'' + 16x = \frac{F_0}{3}\cos(4t)$$

$$t\cos(4t)(-16A + 16A) + t\sin(4t)(-16B + 16B)$$

$$+\cos(4t)(8B) + \sin(4t)(-8A)$$

$$= \frac{F_0}{3}\cos(4t) + 0\cdot\sin(4t)$$

$$8B\cos 4t - 8A\sin 4t = \frac{F_0}{3}\cos(4t) + 0\cdot\sin(4t)$$

$$8B = \frac{F_0}{3}, \quad -8A = 0 \implies B = \frac{F_0}{24}, \quad A = 0$$

$$x_p = \frac{F_0}{24} t\sin 4t$$

$$x = c_1\cos(4t) + c_2\sin 4t + \frac{F_0}{24} t\sin 4t$$

App'ly $x(0) = x'(0) = 0$.

$$x' = -4c_1 \sin 4t + 4c_2 \cos 4t + \frac{F_0}{24} \sin 4t + \frac{F_0}{6} t \cos 4t$$

$$x(0) = c_1 \cos 0 + c_2 \sin 0 + \frac{F_0}{24}(0) \sin 0 = c_1 = 0$$

$$x'(0) = -4c_1 \sin 0 + 4c_2 \cos 0 + \frac{F_0}{24} \sin 0 + \frac{F_0}{6}(0) \cos(0) = 4c_2$$

$$4c_2 = 0 \Rightarrow c_2 = 0$$

The position is

$$\boxed{x(t) = \frac{F_0}{24} t \sin 4t}$$