October 28 Math 2306 sec. 52 Fall 2022

Section 15: Shift Theorems

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

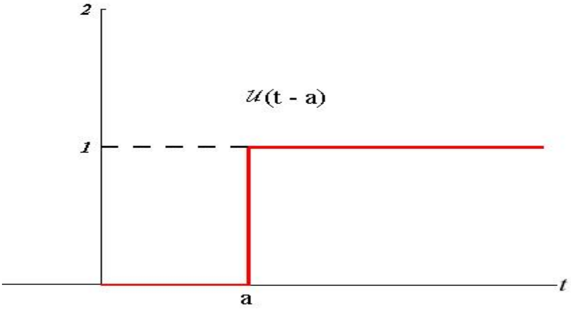

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

We can use the unit step function to write piecewise defined functions in a format convenient for taking Laplace transforms. For example, suppose $0 < a < b < \infty$ and

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t), & a \le t < b \\ f_3(t), & b \le t < \infty \end{cases}$$

We can write f in the form

$$f(t) = f_1(t) - f_1(t)\mathscr{U}(t-a) + f_2(t)\mathscr{U}(t-a) - f_2(t)\mathscr{U}(t-b) + f_3(t)\mathscr{U}(t-b)$$

$$= f_1(t)\big(1 - \mathscr{U}(t-a)\big) + f_2(t)\big(\mathscr{U}(t-a) - \mathscr{U}(t-b)\big) + f_3(t)\mathscr{U}(t-b)$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

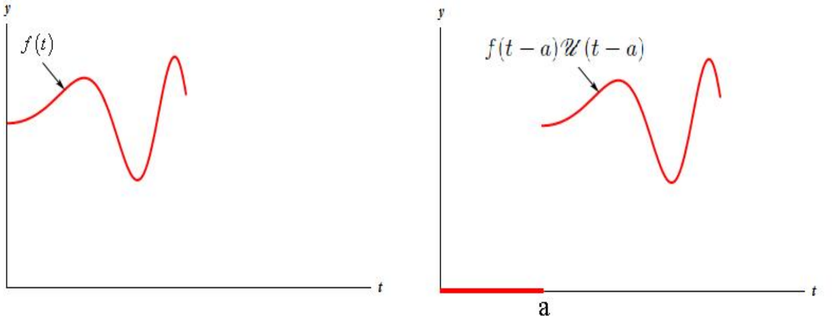

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$ for $a > 0$.

By definition,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \int_0^\infty e^{-st}\mathscr{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\, u(t-a)\,dt + \int_a^\infty e^{-st}\, u(t-a)\,dt$$

$$= \int_0^a e^{-st}\, \cancel{(0)}\,dt + \int_a^\infty e^{-st}\,(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

Convergence requires $s > 0$

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t < \infty \end{cases}$$

$$= \frac{1}{-s} e^{-st} \Big|_a^{\infty}$$

$$= \frac{-1}{s} \left(0 - e^{-s(a)} \right) = \frac{e^{-as}}{s} \qquad s > 0$$

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

A special case is $f(t) = 1$. We just found

$$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = \frac{e^{-as}}{s}.$$

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1 + t)\mathcal{U}(t-1)$$

$$f(t) = 1 + (t-1)\mathcal{U}(t-1)$$

Note: If $f_1(t) = t$ then $f_1(t-1) = t-1$

Example Continued...

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\mathcal{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{1}{s^2}e^{-1s}$$

$*$ for $f_1(t) = t$, $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2} = F(s)$

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$

Example

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Example: Find $\mathscr{L}\{\cos t\, \mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\frac{\pi}{2})\}$

$$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1^2}\right) = \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\cos(t+\tfrac{\pi}{2}) = \cos t \underbrace{\cos \pi/2}_{0} - \sin t \underbrace{\sin\tfrac{\pi}{2}}_{1} = -\sin t$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$ $= \mathscr{L}^{-1}\left\{ e^{-2s} \dfrac{1}{s(s+1)} \right\}$

$F(s) = \dfrac{1}{s(s+1)}$ we need $f(t) = \mathscr{L}^{-1}\{F(s)\}$

Partial fractions

$\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} \implies 1 = A(s+1) + Bs$

$s=0 \quad 1 = A$

$s=-1 \quad 1 = -B$

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$f(t) = 1 - e^{-t}$$

$a = 2$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{e^{-2s}\frac{1}{s(s+1)}\right\}$$

$$= \left(1 - e^{-(t-2)}\right)\mathscr{U}(t-2)$$