October 29 Math 2306 sec. 54 Fall 2021

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform[1] and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$ using integration by parts to get

$\mathscr{L}\{f(t)\} = F(s)$

$$\begin{aligned}\mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st}f'(t)\,dt\\ &= -f(0)+s\int_0^\infty e^{-st}f(t)\,dt\\ &= sF(s)-f(0).\end{aligned}$$

[1] Assume f is of exponential order c for some c.

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
&\ \ \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Laplace Transforms and IVPs

For constants a, b, and c, take the Laplace transform of both sides of the equation and isolate $\mathscr{L}\{y(t)\} = Y(s)$.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $Y(s) = \mathscr{L}\{y(t)\}$, and $G(s) = \mathscr{L}\{g(t)\}$

Take $\mathscr{L}$ of the ODE

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}.$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = G(s)$$

$$a\left(s^2Y(s)-sy(0)-y'(0)\right)+b\left(sY(s)-y(0)\right)+cY(s)=G(s)$$

Use $y(0)=y_0$ and $y'(0)=y_1$

$$as^2Y(s)-ay_0s-ay_1+bsY(s)-by_0+cY(s)=G(s)$$

$$(as^2+bs+c)Y(s)-ay_0s-ay_1-by_0=G(s)$$

$$(as^2+bs+c)Y(s)=ay_0s+ay_1+by_0+G(s)$$

$$ay''+by'+cy=g(t)$$

as^2+bs+c is the characteristic polynomial for the ODE

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2+bs+c}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Solving IVPs

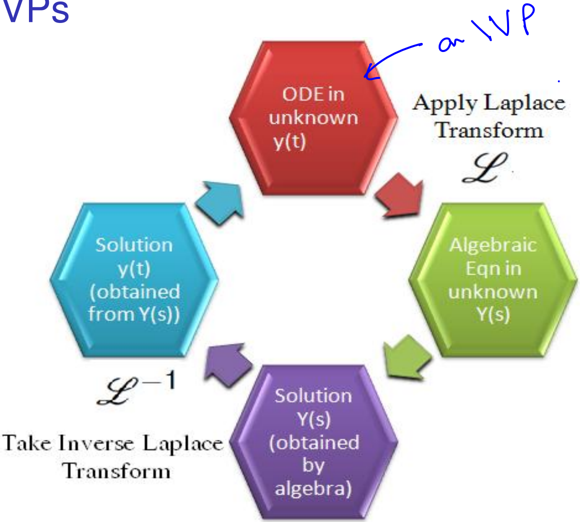

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\} \quad \text{is called the } \textbf{zero input response},$$

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\} \quad \text{is called the } \textbf{zero state response}.$$

Solve the IVP using the Laplace Transform

$$\frac{dy}{dt}+3y=2t, \quad y(0)=2$$

Let $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = 2\left(\frac{1}{s^2}\right)$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2} \quad \Longrightarrow \quad (s+3)Y(s) = \frac{2}{s^2} + 2$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3}$$

We need to decompose $\dfrac{2}{s^2(s+3)}$.

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

clear fractions

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$= As^2 + 3As + Bs + 3B + Cs^2$$

$$2 = (A+C)s^2 + (3A+B)s + 3B$$

$$A + C = 0$$

$$3A + B = 0$$

$$3B = 2 \Rightarrow B = \frac{2}{3}$$

$$A = \frac{-1}{3}B = \frac{-2}{9}, \quad C = -A = \frac{2}{9}$$

$$Y(s) = \frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{2}{9}}{s+3} + \frac{2}{s+3}$$

$$= \frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{20/9}{s+3}$$

The solution to the IVP $y = \mathcal{L}^{-1}\{Y(s)\}$.

$$y(t) = \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}}$$

Solve the IVP using the Laplace Transform

$$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, \quad y'(0) = 0$$

$Y(s) = \mathcal{L}\{y\}$

$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{(s+2)^2}$

$s^2Y - sy(0) - y'(0) + 4\left(sY - y(0)\right) + 4Y = \frac{1}{(s+2)^2}$

use $y(0) = 1$, $y'(0) = 0$

$$s^2Y - s + 4sY - 4 + 4Y = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)Y = \frac{1}{(s+2)^2} + s + 4$$

$$Y = \frac{1}{(s+2)^2(s^2+4s+4)} + \frac{s+4}{s^2+4s+4}$$

Note $s^2+4s+4 = (s+2)^2$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

Note $s+4 = s+2+2$

$$\frac{s+4}{(s+2)^2} = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

Finally

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

If $F(s) = \frac{1}{s^4}$, then $F(s+2) = \frac{1}{(s+2)^4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3!}t^3$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

The solution

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{3!} t^3 e^{-2t} + e^{-2t} + 2 t e^{-2t}$$

$$\boxed{y(t) = \frac{1}{6} t^3 e^{-2t} + e^{-2t} + 2 t e^{-2t}}$$