October 2 Math 2306 sec. 51 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

Consider the linear, homogeneous ODE with constant coefficients,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The characteristic equation for this ODE is the n^{th} degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

Remark: We keep in mind that every fundamental solution set for the n^{th} order ODE will consist of n linearly independent solutions.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solutions for Real Roots

If m is a **real** root of multiplicity k of the characteristic equation, then there are k linearly independent solutions of the ODE

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1} e^{mx}.$$

Solutions for Complex Conjugate Roots

If $\alpha \pm i\beta$ is a **complex conjugate** pair of solutions, each of multiplicity k, of the characteristic equation, then there are $2k$ linearly independent solutions of the ODE

$$e^{\alpha x} \cos(\beta x),\ e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x),\ xe^{\alpha x} \sin(\beta x), \ldots,$$

$$x^{k-1} e^{\alpha x} \cos(\beta x),\ x^{k-1} e^{\alpha x} \sin(\beta x)$$

Example

Consider the ODE

$$y^{(6)} - 6y^{(4)} + 104y''' + 9y'' - 312y' + 2704y = 0.$$

The characteristic equation is

$$m^6 - 6m^4 + 104m^3 + 9m^2 - 312m + 2704 = 0$$

which factors as

$$((m-2)^2 + 9)^2(m+4)^2 = 0.$$

Find the general solution.

Our fundamental solution set must have
six lin. ind. solution in it.

$$((m-2)^2+9)^2(m+4)^2 = 0$$

$((m-2)^2+9)^2 = 0$ or $(m+4)^2 = 0$

If $(m+4)^2 = 0$ then $m+4=0 \Rightarrow m=-4$ is a double root.

two solutions are

$y_1 = e^{-4x}$ and $y_2 = xe^{-4x}$

$((m-2)^2+9)^2 = 0 \Rightarrow (m-2)^2+9=0$

$(m-2)^2 = -9 \Rightarrow m-2 = \pm\sqrt{-9} = \pm 3i$

$m = 2 \pm 3i$ are double roots

$\alpha = 2, \quad \beta = 3$

$$y_3 = e^{2x}\cos(3x), \quad y_4 = e^{2x}\sin(3x)$$

$$y_5 = xe^{2x}\cos(3x), \quad y_6 = xe^{2x}\sin(3x)$$

The general solution is

$$y = c_1 e^{-4x} + c_2 x e^{-4x} + c_3 e^{2x}\cos(3x) + c_4 e^{2x}\sin(3x) + c_5 x e^{2x}\cos(3x) + c_6 x e^{2x}\sin(3x)$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e.g., e^{mx}, m- constant
- sines and/or cosines, $\cos(kx)$ or $\sin(kx)$ k- constant
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left is constant coef., and the right $g(x) = 8x+1$ is a 1st degree polynomial. We will assume the form of y_p and let the ODE fill in the specs. Since g is 1st degree polynomial, maybe y_p is too. Set

$$y_p = Ax + B \quad \text{for some constants}$$

A and B.

[1] **We're only ignoring the y_c part to illustrate the process.**

$$y'' - 4y' + 4y = 8x + 1$$

Set $y_p = Ax + B$ sub it in.

$y_p' = A$

$y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

Matching like terms

$$4A = 8$$

$$-4A + 4B = 1$$

$A = 2, \qquad 4B = 1 + 4A = 1 + 4(2) = 9$

$$B = \frac{9}{4}$$

We found a particular solution

$$\boxed{y_p = 2x + \frac{9}{4}}$$

Find y_c: $\quad y_c'' - 4y_c' + 4y_c = 0$

$$m^2 - 4m + 4 = 0 \implies (m-2)^2 = 0 \quad m = 2$$

double root

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

The gen. solution $\quad y = c_1 e^{2x} + c_2 x e^{2x} + 2x + \frac{9}{4}$