October 2 Math 2306 sec. 53 Fall 2024

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \quad \text{with } a \neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant})\, y' + (\text{constant})\, y?$$

$y = e^{rx}$ $y = \sin(kx)$ or $\cos(kx)$

r - constant   polynomials

[1] We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{rx}$ with r constant.

$$ay'' + by' + cy = 0$$

$y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

sub $a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0$

$e^{rx}(ar^2 + br + c) = 0$

Since $e^{rx} > 0$, this will be true if

$ar^2 + br + c = 0$

We'll have solution(s) e^{rx} if

r is a solution to the quadratic

equation $ar^2+br+c=0$

Suppose a, b, and c are real numbers and $a \neq 0$. The function $y = e^{rx}$ solves the second order, homogeneous ODE

$$ay'' + by' + cy = 0$$

on $(-\infty, \infty)$ provided r is a solution of the quadratic equation

$$ar^2 + br + c = 0.$$

Characteristic (a.k.a. Auxiliary) Equation

The characteristic equation for the second order, linear, homogeneous ODE $ay'' + by' + cy = 0$ is the quadratic equation

$$ar^2 + br + c = 0$$

There are three cases that we must consider.

I $b^2 - 4ac > 0$ then there are two distinct real roots $r_1 \neq r_2$

II $b^2 - 4ac = 0$ then there is one repeated real root $r_1 = r_2 = r$

III $b^2 - 4ac < 0$ then there are two roots that are complex conjugates $r_{1,2} = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots r_1 and r_2. A fundamental solution set consists of

$$y_1 = e^{r_1 x} \quad \text{and} \quad y_2 = e^{r_2 x}.$$

The general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The ODE is 2^{nd} order, linear, homogeneous w/ constant coefficients. The characteristic equation is

$$r^2 - 2r - 2 = 0$$

Complete the square

$$r^2 - 2r + 1 = 2 + 1$$

$$(r-1)^2 = 3$$

$$r - 1 = \pm\sqrt{3}$$

$$\Rightarrow \quad r = 1 \pm \sqrt{3} \, . \quad \text{two distinct roots}$$

$$r_1 = 1+\sqrt{3} \, , \quad r_2 = 1-\sqrt{3}$$

$$y_1 = e^{(1+\sqrt{3})x} \, , \quad y_2 = e^{(1-\sqrt{3})x}$$

The general solution

$$\boxed{y = C_1 e^{(1+\sqrt{3})x} + C_2 e^{(1-\sqrt{3})x}}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $r = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$

$P(x) = \frac{b}{a}$, $\quad -\int P(x)\,dx = -\int \frac{b}{a}\,dx = \frac{-b}{a}x$

$$e^{-\int p(x)\,dx} = e^{-\frac{b}{a}x}, \quad y_1^2 = \left(e^{-\frac{b}{2a}x}\right)^2 = e^{2\left(-\frac{b}{2a}x\right)}$$

$$= e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x\, e^{-\frac{b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root r, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{rx} \quad \text{and} \quad y_2 = xe^{rx}.$$

The general solution is

$$y = c_1 e^{rx} + c_2 xe^{rx}.$$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

The ODE is 2^{nd} order, linear, homogeneous w/ constant coef. The characteristic eqn is

$$r^2 + 6r + 9 = 0$$

$$\Rightarrow \quad (r+3)^2 = 0 \quad \Rightarrow \quad r = -3 \quad \text{Double root}$$

$$y_1 = e^{-3x}, \quad y_2 = x e^{-3x}$$

The general solution $y = c_1 e^{-3x} + c_2 x e^{-3x}$

Apply $y(0) = 4$ and $y'(0) = 0$.

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^{0} + c_2 (0) e^{0} = 4 \Rightarrow c_1 = 4$$

$$y'(0) = -3c_1 e^{0} + c_2 e^{0} - 3c_2 (0) e^{0} = 0$$

$$-3c_1 + c_2 = 0 \Rightarrow c_2 = 3c_1 = 12$$

The solution to the IVP is

$$y = 4e^{-3x} + 12x e^{-3x}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

$\beta > 0$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[2] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$\theta = \beta x$

$$Y_1 = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x)$$

Let $y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

$y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

[2] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$y_1 = e^{\alpha x} \cos(\beta x) \quad , \quad y_2 = e^{\alpha x} \sin(\beta x)$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and $\beta > 0$ be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Find the general solution of $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0.$$

The ODE is linear, homogeneous, with constant coefficients. The characteristic equation is

$$r^2 + 4r + 6 = 0$$

Using the quadratic formula

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$= \frac{-4 \pm \sqrt{8}\, i}{2} = \frac{-4 \pm 2\sqrt{2}\, i}{2} = -2 \pm \sqrt{2}\, i$$

Complex roots w/ $\alpha = -2$ and $\beta = \sqrt{2}$

$$x_1 = e^{-2t} \cos(\sqrt{2}\, t), \quad x_2 = e^{-2t} \sin(\sqrt{2}\, t)$$

The general solution is

$$\boxed{x = c_1 e^{-2t} \cos(\sqrt{2}\, t) + c_2 e^{-2t} \sin(\sqrt{2}\, t)}$$