October 30 Math 2306 sec. 51 Spring 2023

Section 13: The Laplace Transform

Definition:

Let $f(t)$ be piecewise continuous on $[0, \infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. = F(s)$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Remark: We're going to use the Laplace transform as a tool for solving IVPs where the initial conditions are given at $t = 0$.

Example: $f(t) = e^{at}$, $0 \le t < \infty$

Find the Laplace transform of f.

a is any real number

By definition

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$e^{-st} \cdot e^{at} = e^{-st+at} = e^{(-s+a)t} = e^{(a-s)t}$$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$\int e^{kt}\, dt = \frac{1}{k} e^{kt} + C$$

assume $s \neq a$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

assume $a-s<0$

$$= \frac{1}{a-s}\left(0 - e^{0}\right)$$

If $a-s>0$ the integral diverges

$$= \frac{-1}{a-s} = \frac{1}{s-a}$$

What if $a-s=0$ $\quad \int_0^\infty 1\,dt$ is divergent

So $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ if $s>a$

$a-s<0 \Rightarrow a<s$

Computing Laplace Transforms

Despite the definition, Laplace transforms are rarely evaulated by actually integrating. Transforms of common functions (and some *not-so-common*) are extensively cataloged. Tables of transforms are used in practice.

Remark: Googling *table of Laplace transforms* will yield thousands of free webpages and pdfs. The table I'll provide during exams is posted in D2L (and the course page, and the workbook).

Figure: Table of Laplace transforms t-shirt.

A Small Table of Laplace Transforms

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$

- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$

- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(a) $f(t) = \cos(\pi t)$

Use $\mathcal{L}\{\cos(kt)\}$ w/ $k = \pi$

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2 + \pi^2}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\left(\frac{4!}{s^5}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{2(4!)}{s^5} - \frac{1}{s+5} + \frac{3}{s} \quad \text{for } s > 0$$

$\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$s > 0$ $\qquad s > a$ $\qquad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$$\mathcal{L}\{(2-t)^2\} = \mathcal{L}\{4 - 4t + t^2\}$$

$$= 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^2}\right) + \frac{2!}{s^3}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Helpful Tip

If it's not immediately obvious how to evaluate $\mathscr{L}\{f(t)\}$, it's almost always helpful to consider the question

$$\text{How would I evaluate} \quad \int f(t)\,dt?$$

The algebra and function identities used to evaluate integrals are *usually* helpful for evaluating Laplace tranforms.

Examples: Evaluate

How would you evaluate $\int \sin^2(5t)\,dt$

Use $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

(d) $\mathscr{L}\{\sin^2 5t\}$

$$= \mathscr{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\}$$

$$= \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s}{s^2+10^2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s}{s^2+100}$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Exponential Order

Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Being *of exponential order c* means that f doesn't blow up at infinity any faster than e^{ct}.

Definition:Piecewise Continuous

A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem:

If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c \geq 0$, then f has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any c is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

We're going to use the Laplace transform to solve IVPs. So in addition to taking a transform to go from a function of t to a function of s, we'll want to go backwards.

Question: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

Inverse Laplace Transform

Let $F(s)$ be a function. An **inverse Laplace transform** of F is a piecewise continuous function $f(t)$ provided $\mathscr{L}\{f(t)\} = F(s)$. We will use the notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{if} \quad \mathscr{L}\{f(t)\} = F(s).$$

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Using a Table

When using a table of Laplace transforms, the expression must match exactly. For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

so

$$\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that $n = 3$, so there must be 3! in the numerator and the power $4 = 3 + 1$ on s.

Remark: The function $F(s)$ often requires some amount of manimpulation to get it to look like a table entry. There are a few common tricks of the trade to taking inverse Laplace transforms.

Find the Inverse Laplace Transform

$$\mathcal{L}\{t^6\} = \frac{6!}{s^7}$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

Note that $\dfrac{1}{s^7} = \dfrac{1}{s^7}\,\dfrac{6!}{6!} = \dfrac{1}{6!}\,\dfrac{6!}{s^7}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\,\mathcal{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9} + \dfrac{1}{s^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{3}\,\dfrac{3}{s^2+3^2}\right\}$

$= \cos(3t) + \dfrac{1}{3}\sin(3t)$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$ Use partial fractions

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$ clear fractions

$$s-8 = A(s-2) + Bs$$

Set $s=0$ $\quad 0-8 = A(0-2)+B(0) \Rightarrow -8=-2A \Rightarrow A=4$

$s=2$ $\quad 2-8 = A(2-2)+B(2) \Rightarrow -6=2B \Rightarrow B=-3$

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$