October 31 Math 2306 sec. 51 Fall 2022

Section 15: Shift Theorems

Translation (Shift) in s.

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Translation (Shift) in t

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

We can also state this as

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Evaluate $\mathscr{L}\{f(t)\}$

$$f(t) = \begin{cases} 0, & 0 \le t < 4 \\ te^{2t}, & 4 \le t \end{cases} = 0 - 0\,\mathcal{U}(t-4) + te^{2t}\,\mathcal{U}(t-4)$$

$$= te^{2t}\,\mathcal{U}(t-4)$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{te^{2t}\,\mathcal{U}(t-4)\}$$

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Let $g(t) = te^{2t}$ we need $g(t+4)$

$g(t+4) = (t+4)e^{2(t+4)}$

$$= t\,e^{2t+8} + 4\,e^{2t+8}$$

$$= e^{8}\, t e^{2t} + 4e^{8}\, e^{2t}$$

Note : $\mathcal{L}\{te^{2t}\} = F(s-2)$ where $F(s) = \mathcal{L}\{t\} = \dfrac{1}{s^2}$

$$\mathcal{L}\{g(t+4)\} = \mathcal{L}\{e^{8}\, te^{2t} + 4e^{8}\, e^{2t}\}$$

$$= e^{8}\, \mathcal{L}\{te^{2t}\} + 4e^{8}\, \mathcal{L}\{e^{2t}\}$$

$$\mathcal{L}\{g(t+4)\} = \frac{e^{8}}{(s-2)^2} + \frac{4e^{8}}{s-2}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^{2t}\mathcal{U}(t-4)\}$$

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

$$\mathcal{L}\{f(t)\} = e^{-4s}\left(\frac{e^{8}}{(s-2)^2} + \frac{4e^{8}}{s-2}\right)$$

$$= e^{-4(s-2)}\left(\frac{1}{(s-2)^2} + \frac{4}{s-2}\right)$$

$$e^{-4s}\cdot e^{8} = e^{-4s+8} = e^{-4(s-2)}$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi}{4}s}}{s^2+6s+10}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\ \frac{s}{s^2+6s+10}\right\}$$

we need $f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+10}\right\} = \mathcal{L}^{-1}\left\{F(s)\right\}$

we'll complete the square on $s^2+6s+10$

$$s^2+6s+9-9+10 = (s+3)^2+1$$

need $s+3$ here

$$\frac{s}{(s+3)^2+1} = \frac{s+3-3}{(s+3)^2+1} = \frac{(s+3)}{(s+3)^2+1} - \frac{3}{(s+3)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-3t}\cos t$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s+3)^2+1}\right\} = 3e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 3e^{-3t}\sin t$$

$$\mathcal{L}^{-1}\left\{F(s)\right\} = e^{-3t}\cos t - 3e^{-3t}\sin t = f(t)$$

$$\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi}{4}s}}{s^2+6s+10}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\frac{s}{s^2+6s+10}\right\}$$

$$= f(t-\pi/4)\,u(t-\pi/4)$$

$$= \left(e^{-3(t-\pi/4)}\cos(t-\pi/4) - 3e^{-3(t-\pi/4)}\sin(t-\pi/4)\right)u(t-\pi/4)$$