October 31 Math 2306 sec. 52 Fall 2022

Section 15: Shift Theorems

Translation (Shift) in s.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Translation (Shift) in t

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

We can also state this as

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Evaluate $\mathscr{L}\{f(t)\}$

$$f(t) = \begin{cases} 0, & 0 \le t < 4 \\ te^{2t}, & 4 \le t \end{cases} = 0 - 0\,\mathscr{U}(t-4) + te^{2t}\,\mathscr{U}(t-4)$$

$$= te^{2t}\,\mathscr{U}(t-4)$$

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Let $g(t) = te^{2t}$, i.e. need $g(t+4)$

$$g(t+4) = (t+4)e^{2(t+4)} = te^{2(t+4)} + 4e^{2(t+4)}$$

$$= e^{8}te^{2t} + 4e^{8}e^{2t}$$

$$\mathscr{L}\{te^{2t}\} = F(s-2) \text{ where } F(s) = \mathscr{L}\{t\} = \frac{1}{s^2}$$

$$\mathscr{L}\{g(t+4)\} = \mathscr{L}\{e^8 te^{2t} + 4e^8 e^{2t}\}$$

$$= e^8 \mathscr{L}\{te^{2t}\} + 4e^8 \mathscr{L}\{e^{2t}\}$$

$$= \frac{e^8}{(s-2)^2} + \frac{4e^8}{s-2}$$

$$f(t) = te^{2t}\mathscr{U}(t-4)$$

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

$$\mathcal{L}\{f(t)\} = e^{-4s}\left(\frac{e^8}{(s-2)^2} + \frac{4e^8}{s-2}\right)$$

Note $$e^{-4s}e^{8} = e^{-4s+8} = e^{-4(s-2)}$$

$$\mathcal{L}\{f(t)\} = e^{-4(s-2)}\left(\frac{1}{(s-2)^2} + \frac{4}{s-2}\right)$$

Evaluate

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

$$\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi}{4}s}}{s^2+6s+10}\right\} = \mathscr{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\ \frac{s}{s^2+6s+10}\right\}$$

$F(s) = \dfrac{s}{s^2+6s+10}$ need $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

need $s+3$

$$s^2+6s+9-9+10 = (s+3)^2+1$$

$$F(s) = \frac{s}{(s+3)^2+1} = \frac{s+3-3}{(s+3)^2+1} = \frac{s+3}{(s+3)^2+1} - \frac{3}{(s+3)^2+1}$$

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

$$\mathscr{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-3t}\cos t$$

$$\mathscr{L}^{-1}\left\{\frac{3}{(s+3)^2+1}\right\} = 3e^{-3t}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 3e^{-3t}\sin t$$

So $f(t) = \mathscr{L}^{-1}\{F(s)\} = e^{-3t}\cos t - 3e^{-3t}\sin t$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

$$\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi}{4}s}}{s^2+6s+10}\right\} = \mathscr{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\ \frac{s}{s^2+6s+10}\right\}$$

$$= \left(e^{-3(t-\pi/4)}\cos\left(t-\frac{\pi}{4}\right) - 3e^{-3(t-\pi/4)}\sin(t-\pi/4)\right)\mathscr{U}(t-\pi/4)$$