October 3 Math 2306 sec. 51 Fall 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Basics of the Method

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

- Confirm the left is constant coefficient, and g is from the right class of functions.
- **Find y_c first.**
- The principle of superposition can be used if the right side is a sum of different kinds of g's.
- Identify what **type** of function g is.
- Assume y_p is the same **type**.
- Compare assume y_p to y_c and modify (multiply by x^n) if needed.
- Remember that polynomials include all decending powers, and sines and cosines go together.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

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Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: y_c solves $y'' - 4y' + 4y = 0$

Characteristic eqn. $m^2 - 4m + 4 = 0$

Find roots: $(m-2)^2 = 0 \Rightarrow m = 2$ double root

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$ $\quad y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let's break this up and let y_{p_1} solve

$y'' - 4y' + 4y = \sin(4x)$ $\quad g_1(x) = \sin(4x)$

and y_{p_2} solve $y'' - 4y' + 4y = xe^{2x}$ $g_2(x) = xe^{2x}$

For $g_1(x) = \sin 4x$,

$$\boxed{y_{p_1} = A\sin(4x) + B\cos(4x)}$$

This has no like terms in common with y_c. It is correct

For $g_2(x) = xe^{2x}$

$y_{p_2} = (Cx + D)e^{2x}$ not correct

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$ Has like terms in common w/ y_c

$$y_{p_2} = (Cx+D)e^{2x} \cdot x$$

$$= (Cx^2+Dx)e^{2x}$$ still not correct,

$$y_{p_2} = (Cx+D)e^{2x} \cdot x^2$$

$$= (Cx^3+Dx^2)e^{2x}$$ This is correct

$$y_{p_2} = Cx^3e^{2x} + Dx^2e^{2x}$$

The form of y_p is

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4 - 4x^2 + 1$$

Find y_c: $y''' - y'' + y' - y = 0$

Characteristic eqn: $m^3 - m^2 + m - 1 = 0$

factor. $m^2(m-1) + (m-1) = 0$

$(m^2+1)(m-1) = 0$

$m - 1 = 0 \Rightarrow m = \underline{1}$ real root

$m^2+1 = 0 \Rightarrow m^2 = -\underline{1} \Rightarrow m = \pm i$ complex $\alpha = 0,\ \beta = \underline{1}$

The solutions are

$$y_1 = e^x \quad , \quad y_2 = \cos x \quad , \quad y_3 = \sin x$$

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

$$y''' - y'' + y' - y = \cos x + x^4 - 4x^2 + 1$$

Let $g_1(x) = \cos x$ and $g_2(x) = x^4 - 4x^2 + 1$

and look for $y_p = y_{p_1} + y_{p_2}$

For $g_1(x) = \cos x$

$$y_{p_1} = A\cos x + B\sin x$$

Duplicates part of y_c

$$y_{p_1} = (A\cos x + B\sin x)\,x$$

$= Ax\cos x + Bx\sin x$ This is correct

For $g_2(x) = x^4 - 4x^2 + 1$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$

This is the correct form.

$$\boxed{y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G}$$

This is the form of the particular solution.

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: $y'' - y = 0$

Characteristic eqn is: $m^2 - 1 = 0 \Rightarrow m^2 = 1$

$m = 1$ or $m = -1$

$y_1 = e^x$ and $y_2 = e^{-x}$ (two real roots case)

$$\boxed{y_c = c_1 e^x + c_2 e^{-x}}$$

Find y_p: $g(x) = 4e^{-x}$

$y_p = Ae^{-x}$ duplicates y_c

$y_p = Axe^{-x}$ this is the correct form

$y_p = Axe^{-x}$

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$

Sub into the ODE

$y_p'' - y_p = 4e^{-x}$

$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$

Collect like terms

$xe^{-x}(A - A) + e^{-x}(-2A) = 4e^{-x}$

$-2A = 4 \Rightarrow A = -2$

So $y_p = -2xe^{-x}$

The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} - 2x e^{-x}.$$

Apply the I.C. $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2(0) e^{0} = -1 \implies c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2e^{0} + 2\cdot 0 e^{0} = 1 \implies c_1 - c_2 = 3$$

$$\begin{array}{r} c_1 + c_2 = -1 \\ c_1 - c_2 = 3 \\ \hline 2c_1 = 2 \end{array} \implies c_1 = 1$$

add

$c_2 = -1 - c_1 = -1 - 1 = -2$

$$\boxed{\text{The solution to the IVP is} \quad y = e^{x} - 2e^{-x} - 2xe^{-x}}$$