October 3 Math 2306 sec. 52 Fall 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Basics of the Method

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

- Confirm the left is constant coefficient, and g is from the right class of functions.
- **Find y_c first.**
- The principle of superposition can be used if the right side is a sum of different kinds of g's.
- Identify what **type** of function g is.
- Assume y_p is the same **type**.
- Compare assume y_p to y_c and modify (multiply by x^n) if needed.
- Remember that polynomials include all decending powers, and sines and cosines go together.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

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Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $\quad y'' - 4y' + 4y = 0$

Characteristic equation

$$m^2 - 4m + 4 = 0$$

Find roots $\quad (m-2)^2 = 0 \Rightarrow m = 2$ double root

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = xe^{2x}$$

Let's use superposition to break this into two problems

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Let y_{p_1} solve $y'' - 4y' + 4y = \sin(4x)$ and

Let y_{p_2} solve $y'' - 4y' + 4y = xe^{2x}$

For $g_1(x) = \sin(4x)$ set

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$

This doesn't share like terms in common with y_c, so it is correct

The fundamental solution set is

$$\boxed{y_1 = e^{2x} \quad \text{and} \quad y_2 = xe^{2x}}$$

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

For $g_2(x) = x e^{2x}$

$$y_{p_2} = (Cx + D) e^{2x}$$

not correct duplicates y_c

$$y_{p_2} = (Cx + D) e^{2x} \cdot x$$

$$= (Cx^2 + Dx) e^{2x}$$

xe^{2x} still duplicates part of y_c

$$y_{p_2} = (Cx + D) e^{2x} \cdot x^2$$

$$= (Cx^3 + Dx^2) e^{2x}$$

This is correct

Finally,

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3 e^{2x} + Dx^2 e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4 - 4x^2 + 1$$

Find y_c: $\quad y''' - y'' + y' - y = 0$

Characteristic eqn: $\quad m^3 - m^2 + m - 1 = 0$

factor $\quad m^2(m-1) + (m-1) = 0$

$$(m^2+1)(m-1) = 0$$

$m - 1 = 0 \implies m = 1$ real root

$m^2 + 1 = 0 \implies m^2 = -1 \implies m = \pm i$ complex $\alpha = 0$ and $\beta = 1$.

The fundamental solution set is

$y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

$$y''' - y'' + y' - y = \cos x + x^4 - 4x^2 + 1$$

Let $g_1(x) = \cos x$

$y_{p_1} = A\cos x + B\sin x$ Duplicates part of y_c.

$y_{p_1} = (A\cos x + B\sin x)x$

$= Ax\cos x + Bx\sin x$ this is correct

Let $g_2(x) = x^4 - 4x^2 + 1$

$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

this is correct

Finally, the form of the particular solution is

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: $y'' - y = 0$

Characteristic eqn is: $m^2 - 1 = 0 \Rightarrow m^2 = 1$

$m = 1$ or $m = -1$

$y_1 = e^{x}$ and $y_2 = e^{-x}$ (two real roots case)

$$\boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$$

Find y_p: $g(x) = 4e^{-x}$

$y_p = Ae^{-x}$ duplicates y_c

$y_p = Axe^{-x}$ this is the correct form

$$y_p = Axe^{-x}$$
$$y_p' = Ae^{-x} - Axe^{-x}$$
$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$$

Sub into the ODE

$$y_p'' - y_p = 4e^{-x}$$
$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

Collect like terms

$$xe^{-x}(A - A) + e^{-x}(-2A) = 4e^{-x}$$
$$-2A = 4 \Rightarrow A = -2$$

So $y_p = -2xe^{-x}$

The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x}$$

Apply the I.C. $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2(0) e^{0} = -1 \Rightarrow c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2e^{0} + 2 \cdot 0 e^{0} = 1 \Rightarrow c_1 - c_2 = 3$$

$$\begin{array}{l} c_1 + c_2 = -1 \\ c_1 - c_2 = 3 \\ \hline \end{array}$$

add

$$2c_1 = 2 \Rightarrow c_1 = 1$$

$c_2 = -1 - c_1 = -1 - 1 = -2$

The solution to the IVP is

$$y = e^{x} - 2e^{-x} - 2xe^{-x}$$