October 4 Math 2306 sec. 51 Fall 2021

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

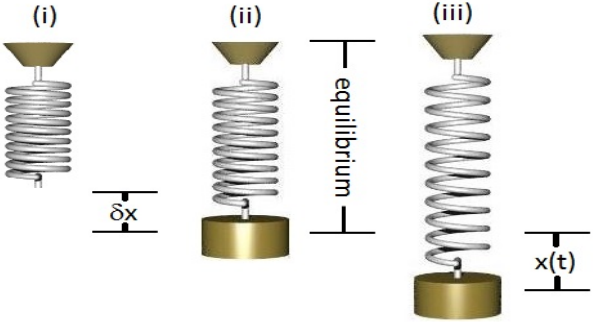

Figure: We track the displacement x from the equilibrium position. Recall δx is the displacement **in** equilibrium. We assume there are no damping nor external forces (for now).

Simple Harmonic Motion

We derive the equation for displacement. Given initial displacement x_0 and initial velocity x_1,

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1}$$

where $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion**[1].

k-spring constant, *m*-object's mass, *g*-acceleration due to gravity.

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[1] **Caution:** The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[2]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[2] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin \phi = \frac{x_0}{A}, \quad \text{with} \quad \cos \phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos\hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

No driving + no damping $\Rightarrow$

Simple harmonic motion.

The ODE is

$$x'' + \omega^2 x = 0$$

or $mx'' + kx = 0$ $\quad \omega^2 = \frac{k}{m}$

We need to find ω^2.

We have displacement in equilibrium

$$\delta x = 6 \text{ in}$$

We have $\omega^2 = \dfrac{g}{\delta x}$

$g = 32\ ft/sec^2$. We can convert δx to feet

$$\delta x = 6 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) = \frac{1}{2} \text{ ft}$$

$$\omega^2 = \frac{g}{\delta x} = \frac{32\ ft/sec^2}{\frac{1}{2}\ ft} = 64 \frac{1}{sec^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

The ODE will be $x'' + \omega^2 x = 0$

Note: $\delta x = 6$ in so $\omega^2 = 64 \frac{1}{\text{sec}^2}$

Given the weight, we can find m and k. The weight of the object

$W = 4$ lb

The mass $m = \frac{W}{g} = \frac{4\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{8}$ slugs.

The spring constant $k = \frac{W}{\delta x} = \frac{4\text{ lb}}{\frac{1}{2}\text{ ft}} = 8\ \frac{\text{lb}}{\text{ft}}$

So again $\omega^2 = \frac{k}{m} = \frac{8\ \frac{\text{lb}}{\text{ft}}}{\frac{1}{8}\text{ slugs}} = 64\ \frac{1}{\text{sec}^2}$

The IVP is

4 ft above equilibrium

$$x'' + 64x = 0 \qquad x(0) = 4 \qquad x'(0) = -24$$

downward velocity 24 ft/sec

Find the general solution to the ODE.

Let's use r for the characteristic polynomial.

$$r^2 + 64 = 0$$

$$r^2 = -64 \Rightarrow r = \pm 8i = 0 \pm 8i$$

$$x_1 = e^{0t} \cos(8t), \quad x_2 = e^{0t} \sin(8t)$$

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

Apply $x(0) = 4$, $x'(0) = 24$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \underset{1}{\cos(0)} + c_2 \underset{0}{\sin(0)} = 4 \implies c_1 = 4$$

$$x'(0) = -8c_1 \sin(0) + 8c_2\cos(0) = -24$$

$$8c_2 = -24 \implies c_2 = -3$$

The equation of motion is

$$\boxed{x(t) = 4\cos(8t) - 3\sin(8t)}$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

The frequency $f = \frac{1}{T} = \frac{4}{\pi}$

The amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5$$

If $\quad x(t) = A \sin(\omega t + \phi)$

$$= 5 \sin(8t + \phi)$$

where $\quad \sin\phi = \frac{4}{5}, \quad \cos\phi = \frac{-3}{5}$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21 \qquad \phi \approx 127^\circ$$

The range for $\sin^{-1} q$

is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

The range for $\cos^{-1} q$

is $[0, \pi]$.

Free Damped Motion

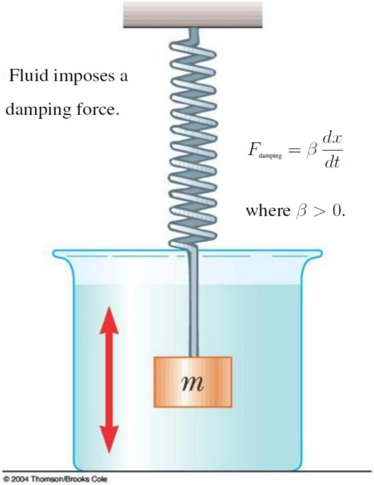


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

$$\text{Total Force} = \text{Force of spring} + \text{Force of damping}$$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

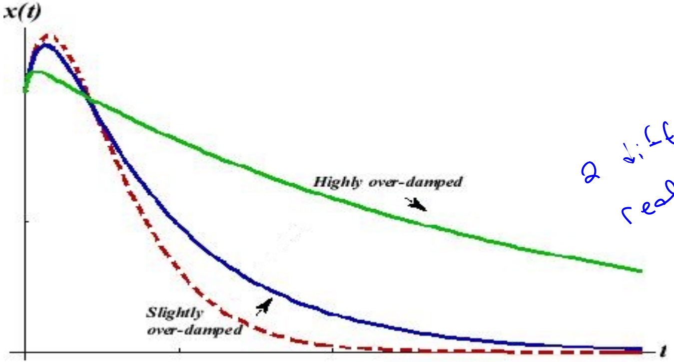
$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$



2 different real roots

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

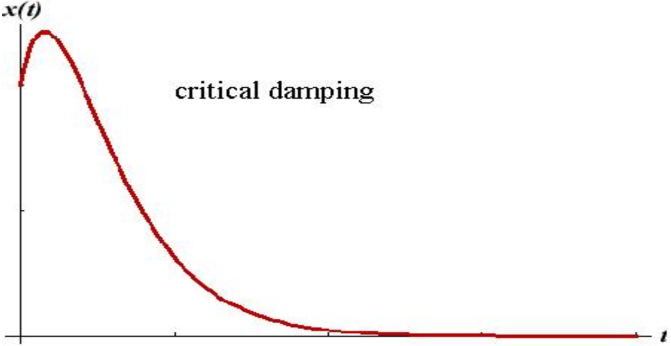


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

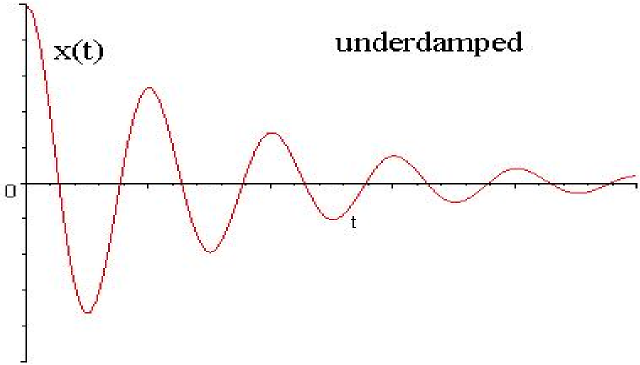


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

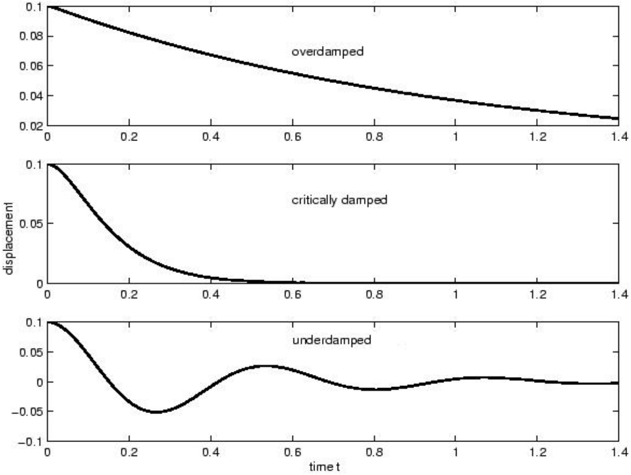


Figure: Comparison of motion for the three damping types.