October 4 Math 2306 sec. 51 Fall 2024

Section 8: Homogeneous Equations with Constant Coefficients

We are considering linear, homogeneous equations with constant coefficients.

Second Order Case

Suppose a, b, and c are real numbers and $a \neq 0$. The function $y = e^{rx}$ solves the second order, homogeneous ODE

$$ay'' + by' + cy = 0$$

on $(-\infty, \infty)$ provided r is a solution of the quadratic equation

$$ar^2 + br + c = 0.$$

This polynomial equation is called the **characteristic equation**.

Second Order Cases $ay'' + by' + cy = 0$

Two Distinct Real Roots

If $ar^2 + br + c = 0$ has two different real roots, r_1 and r_2, then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ form a fundamental solution set. The general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

One Repeated Real Root

If $ar^2 + br + c = 0$ has one repeated real root, r, then $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ form a fundamental solution set. The general solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Second Order Cases $ay'' + by' + cy = 0$

Complex Conjugate Roots

If $ar^2 + br + c = 0$ complex roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, with α and β real numbers and $\beta > 0$, then

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x)$$

form a fundamental solution set. The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Solve the initial value problem

$$4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

The ODE is linear, 2^{nd} order, homogeneous w/ constant coef. The characteristic eqn is

$$4r^2 + 4r + 1 = 0$$

$$(2r+1)^2 = 0 \implies r = -\frac{1}{2} \text{ Double root}$$

$$y_1 = e^{-\frac{1}{2}x}, \quad y_2 = x e^{-\frac{1}{2}x}$$

The general solution to the ODE

$$y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-1/2 x}$$

Apply $y(0) = 1$ and $y'(0) = -1$.

$$y' = -\frac{1}{2} c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} - \frac{1}{2} c_2 x e^{-\frac{1}{2}x}$$

$$y(0) = c_1 e^0 + c_2(0) e^0 = 1 \implies c_1 = 1$$

$$y'(0) = -\frac{1}{2} c_1 e^0 + c_2 e^0 - \frac{1}{2} c_2 (0) e^0 = -1$$

$$-\frac{1}{2} c_1 + c_2 = -1$$

$$c_2 = -1 + \frac{1}{2} c_1$$

$$c_2 = -\frac{1}{2}$$

The solution to the IVP is

$$y = e^{-\frac{1}{2}x} - \frac{1}{2}x e^{-\frac{1}{2}x}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.
- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, r may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root r is repeated k times, we get k linearly independent solutions
$$e^{rx}, \quad xe^{rx}, \quad x^2 e^{rx}, \quad \ldots, \quad x^{k-1} e^{rx}$$
or in conjugate pairs cases $2k$ solutions
$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$

The ODE is 3^{rd} order, linear, homogeneous, w/ constant coef. The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0 \Rightarrow r = 1 \quad \text{triple root}$$

solutions are

$$y_1 = e^{1x}, \quad y_2 = x e^{1x}, \quad y_3 = x^2 e^{1x}$$

The general solution is

$$y = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x}$$

Example

Find the general solution of the ODE.

$y^{(4)} + 3y'' - 4y = 0$

The ODE is linear, homogeneous, w/ constant coef. The characteristic equation is

$$r^4 + 3r^2 - 4 = 0$$

$$(r^2 + 4)(r^2 - 1) = 0$$

$$(r^2 + 4)(r-1)(r+1) = 0$$

$\Rightarrow r = 1, \; r = -1 \qquad r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$

$\alpha \pm i\beta = \pm 2i \Rightarrow \alpha = 0 \quad \beta = 2$

$y_1 = e^{1x}$, $y_2 = e^{-1x}$,

$y_3 = e^{0x} \cos(2x)$, $y_4 = e^{0x} \sin(2x)$

$= \cos(2x)$ $\qquad = \sin(2x)$

The general solution

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)}$$

Example

Consider the ODE

$$y^{(6)} - 6y^{(4)} + 104y''' + 9y'' - 312y' + 2704y = 0.$$

The characteristic equation is

$$r^6 - 6r^4 + 104r^3 + 9r^2 - 312r + 2704 = 0$$

which factors as

$$((r-2)^2 + 9)^2 (r+4)^2 = 0.$$

Find the general solution.

Find the roots.

$(r+4)^2 = 0 \Rightarrow r+4=0 \Rightarrow r=-4$ Double root

$y_1 = e^{-4x}$, $y_2 = x e^{-4x}$

$$((r-2)^2+9)^2(r+4)^2=0$$

$$((r-2)^2+9)^2=0 \Rightarrow (r-2)^2+9=0$$

$$(r-2)^2=-9$$

$$r-2=\pm\sqrt{-9}=\pm 3i$$

$$r=2\pm 3i$$ each a double root.

$\alpha=2 \quad \beta=3$

$y_3=e^{2x}\cos(3x)$; $y_4=e^{2x}\sin(3x)$

$y_5=xe^{2x}\cos(3x)$, $y_6=xe^{2x}\sin(3x)$.

The general solution

$$y = e^{-4x}(c_1 + c_2 x) + e^{2x}\cos(3x)(c_3 + c_5 x)$$

$$+ e^{2x}\sin(3x)(c_4 + c_6 x)$$