October 4 Math 2306 sec. 51 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

The general solution, $y = y_c + y_p$ will require both y_c and y_p. The associated homogeneous equation will be constant coefficient, so we use the method of the last section to find y_c.

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

by

- guessing that y_p is the same kind of function as g,
- setting it up with **undetermined** coefficients, $y_p = Ax + B$, and
- substituting it into the ODE to find the coefficients, A and B, that work.

The complementary solution solves $y'' - 4y' + 4y = 0$. Using the characteristic equation, that turns out to be $y_c = c_1 e^{2x} + c_2 x e^{2x}$. Making the general solution

$$y = c_1 e^{2x} + c_2 x e^{2x} + 2x + \frac{9}{4}.$$

The Method: Assume y_p has the same **form** as $g(x)$

Find a particular solution to $\quad y'' - 4y' + 4y = 6e^{-3x}$

The left is constant coef and the right is an exponential.

$g(x) = 6\,e^{-3x}$ (a constant times e^{-3x})

Set $y_p = Ae^{-3x}$ for some constant A.

substitute into the ODE.

$y_p = Ae^{-3x}$, $y_p' = -3Ae^{-3x}$, $y_p'' = 9Ae^{-3x}$

$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

These match if $25A = 6 \Rightarrow A = \frac{6}{25}$

We found a particular solution

$$y_p = \frac{6}{25}e^{-3x}$$

Check: $y_p = \frac{6}{25}e^{-3x}$, $y_p' = \frac{-18}{25}e^{-3x}$, $y_p'' = \frac{54}{25}e^{-3x}$

$$y_p'' - 4y_p' + 4y_p = \frac{54}{25}e^{-3x} - 4\left(\frac{-18}{25}e^{-3x}\right) + 4\left(\frac{6}{25}e^{-3x}\right) = 6e^{-3x}$$

$$\begin{array}{r} {}^{1}\\ 54 \\ 72 \\ 24 \\ \hline 150 \end{array}$$

$$\frac{54}{25} + \frac{72}{25} + \frac{24}{25} = \frac{150}{25} = 6$$

The Initial Guess Must Be General in Form

Find a particular solution to $\quad y'' - 4y' + 4y = 16x^2$

The left side is constant coef and the right is a polynomial.

$g(x) = 16x^2$

g is a monomial, a constant times x^2

g is also a 2^{nd} degree polynomial

Suppose we consider $g(x)$ as a monomial and assume y_p is a constant times x^2.

Set $y_p = Ax^2$. Substitute

$y_p' = 2Ax$

$y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2$$

Matching like terms

$$\left.\begin{array}{lll} 4Ax^2 = 16x^2 & \Rightarrow & 4A = 16 \\ -8Ax = 0 & \Rightarrow & -8A = 0 \\ 2A = 0 & \Rightarrow & 2A = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A = 4 \ \underline{\text{and}} \\ A = 0 \end{array}$$

impossible!

The process breaks down since the form for y_p is wrong.

Let's recognize $g(x) = 16x^2$ as a 2^{nd} degree polynomial. Assume y_p is also a 2^{nd} degree poly.

Set $y_p = Ax^2 + Bx + C$. Substitute

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$\underline{4A}\,x^2 + (\underline{-8A + 4B})x + (\underline{2A - 4B + 4C}) = \underline{16}x^2 + \underline{0}x + \underline{0}$$

Match like terms

x^2	$4A$	$= 16$
x	$-8A + 4B$	$= 0$
const	$2A - 4B + 4C$	$= 0$

Top eqn. $A = 4$

middle $4B = 8A \Rightarrow B = 2A = 8$

last eqn $4C = -2A + 4B \Rightarrow C = -\frac{1}{2}A + B = -\frac{1}{2}(4) + 8 = 6$

$A = 4,\ B = 8,\ C = 6$

The particular solution

$$y_p = 4x^2 + 8x + 6$$

The general solution to the ODE is

$$y = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{y_c} + \underbrace{4x^2 + 8x + 6}_{y_p}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

That is, $A = -5$ and $A = 0$.

This is impossible as it would require $-5 = 0$. We failed to account for the cosine.

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Two Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.

Remark: The key point is that we have to account for **ALL** of the terms that might arise from taking derivatives.

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2nd degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3rd degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ (1[st] degree polynomial times e^{3x})

$$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ (two linear combos of sine/cosine)

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$