# October 4 Math 2306 sec. 52 Fall 2021

## Section 11: Linear Mechanical Equations

### Simple Harmonic Motion

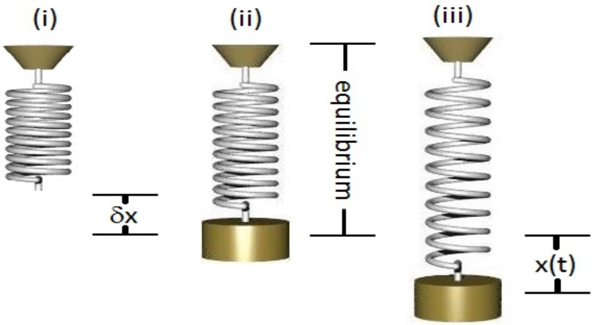



Figure: We track the displacement $x$ from the equilibrium position. Recall $\delta x$ is the displacement **in** equilibrium. We assume there are no damping nor external forces (for now).

# Simple Harmonic Motion

We derive the equation for displacement. Given initial displacement $x_0$ and initial velocity $x_1$,

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1}$$

where $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion**[1].

*k*-spring constant, *m*-object's mass, *g*-acceleration due to gravity.

[1] **Caution:** The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

# Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[2]
- the circular (or angular) frequency $\omega$, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[2] Various authors call $f$ the natural frequency and others use this term for $\omega$.

# Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** $\phi$ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

# Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos\hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

# Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

no driving and no damping $\Longrightarrow$
simple harmonic motion.

The ODE will look like

$$x'' + \omega^2 x = 0$$

or $\quad mx'' + kx = 0 \quad$ with $\frac{k}{m} = \omega^2$

We need to find $\omega^2$.

We're given the displacement in equilibrium $\delta x = 6$ in.

We can use $\omega^2 = \dfrac{g}{\delta x}$

$$g = 32\ \mathrm{ft}/\mathrm{sec}^2 \qquad \delta x = 6\ \mathrm{in}\left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in}}\right) = \frac{1}{2}\ \mathrm{ft}$$

Hence $\omega^2 = \dfrac{32\ \mathrm{ft}/\mathrm{sec}^2}{\frac{1}{2}\ \mathrm{ft}} = 64\ \dfrac{1}{\mathrm{sec}^2}$

The ODE is $\boxed{x'' + 64x = 0.}$

# Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec$^2$.)

The ODE is $mx'' + kx = 0$

which is $x'' + \omega^2 x = 0$ in standard form.

Here $\delta x = 6$ in, so we know that $\omega^2 = 64$.

Let's find $m$ and $k$.

The mass $m = \frac{W}{g}$ where $W$ is the weight of the object. Given $W = 4\text{ lb}$

$$m = \frac{4\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{8}\text{ slugs}$$

The spring constant $k = \frac{W}{\delta x} = \frac{4\text{ lb}}{\frac{1}{2}\text{ ft}} = 8\,\frac{\text{lb}}{\text{ft}}$

The ODE is $\frac{1}{8}x'' + 8x = 0$

$$\Longrightarrow \quad x'' + 64x = 0$$

$x(0) = 4$ $\qquad$ $x'(0) = -24$ $\leftarrow$ downward velocity $24\,\frac{\text{ft}}{\text{sec}}$.

4ft above equilibrium

Solving the ODE:

The characteristic equation is

$$r^2 + 64 = 0$$

$$r^2 = -64 \Rightarrow r = \pm i8 = 0 \pm i8$$

$$x_1 = e^{0t}\cos(8t), \quad x_2 = e^{0t}\sin(8t)$$

The general solution is

$$x(t) = c_1\cos(8t) + c_2\sin(8t)$$

Apply $x(0) = 4$, $x'(0) = -24$

$$x'(t) = -8c_1\sin(8t) + 8c_2\cos(8t)$$

$$x(0) = \underbrace{c_1\cos(0)}_{1} + \underbrace{c_2\sin(0)}_{0} = 4$$

$$\Rightarrow \quad c_1 = 4$$

$$x'(0) = -8c_1\sin(0) + 8c_2\cos(0) = -24$$

$$8c_2 = -24 \Rightarrow c_2 = -3$$

The displacement

$$x(t) = 4\cos(8t) - 3\sin(8t)$$

This is the equation of motion.

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

The frequency $f = \frac{1}{T} = \frac{4}{\pi}$

The amplitude $A = \sqrt{4^2 + (-3)^2} = 5$

The phase shift $\phi$ satisfies

$$\sin\phi = \frac{4}{5}, \quad \cos\phi = \frac{-3}{5}$$

$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$ about $127^\circ$

# Free Damped Motion

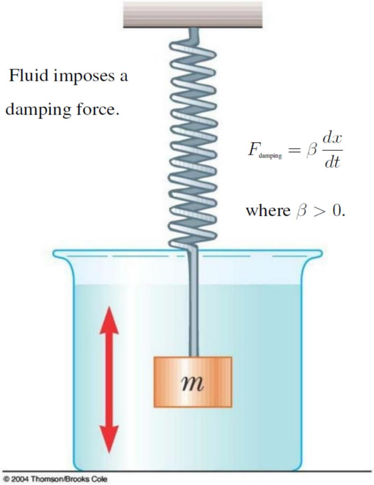



Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

# Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

$$\text{Total Force} = \text{Force of spring} + \text{Force of damping}$$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

# Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

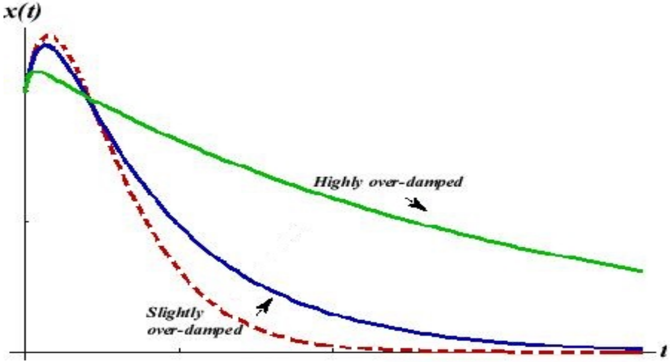



Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

# Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

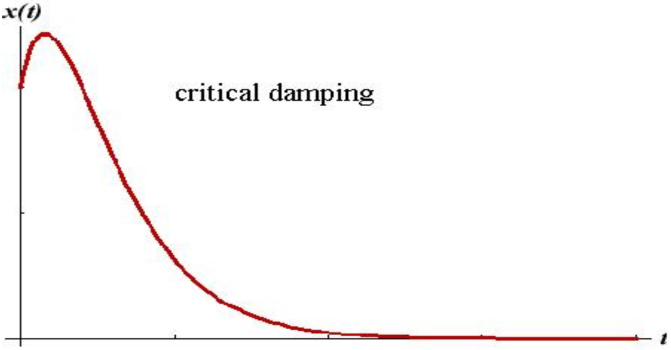



Figure: $\boxed{\text{One real root.}}$ No oscillations. Fastest approach to equilibrium.

# Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

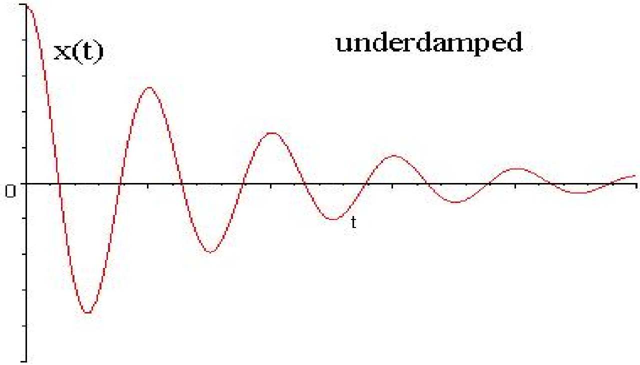
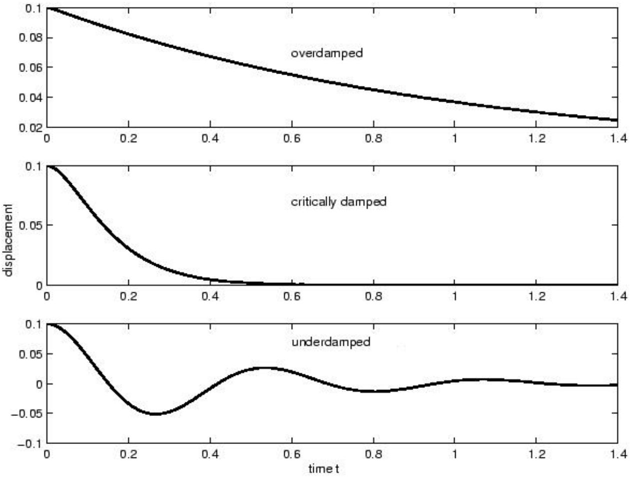



Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

# Comparison of Damping




2 real roots

1 repeated

complex

Figure: Comparison of motion for the three damping types.