October 4 Math 2306 sec. 52 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

The general solution, $y = y_c + y_p$ will require both y_c and y_p. The associated homogeneous equation will be constant coefficient, so we use the method of the last section to find y_c.

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

by

- guessing that y_p is the same kind of function as g,
- setting it up with **undetermined** coefficients, $y_p = Ax + B$, and
- substituting it into the ODE to find the coefficients, A and B, that work.

The complementary solution solves $y'' - 4y' + 4y = 0$. Using the characteristic equation, that turns out to be $y_c = c_1 e^{2x} + c_2 x e^{2x}$. Making the general solution

$$y = c_1 e^{2x} + c_2 x e^{2x} + 2x + \frac{9}{4}.$$

The Method: Assume y_p has the same **form** as $g(x)$

Find a particular solution to $\quad y'' - 4y' + 4y = 6e^{-3x}$

The left is constant coef. and the right is an exponential.

$g(x) = 6e^{-3x}$ a constant times e^{-3x}

Assume $y_p = Ae^{-3x}$ sub this into the ODE

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

Matching like terms $\quad 25A = 6 \Rightarrow A = \frac{6}{25}$

$$\boxed{\text{we found a particular solution } \quad y_p = \frac{6}{25}e^{-3x}}$$

Check: $\quad y_p = \frac{6}{25}e^{-3x}, \quad y_p' = \frac{-18}{25}e^{-3x}, \quad y_p'' = \frac{54}{25}e^{-3x}$

$$y_p'' - 4y_p' + 4y_p = \frac{54}{25}e^{-3x} - 4\left(\frac{-18}{25}e^{-3x}\right) + 4\left(\frac{6}{25}e^{-3x}\right) =$$

$$= \frac{150}{25}e^{-3x} = 6e^{-3x}$$

$$\begin{array}{r} {}^{1}54 \\ 72 \\ 24 \\ \hline 150 \end{array}$$

The Initial Guess Must Be General in Form

Find a particular solution to $\quad y'' - 4y' + 4y = 16x^2$

The ODE is the right type (constant coef left, polynomial right).

$$g(x) = 16x^2$$

$g(x)$ is a monomial, a constant times x^2

$g(x)$ is also a 2^{nd} degree polynomial

Suppose we think of g as a monomial. and set $y_p = Ax^2$ sub this into the ODE

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2$$

Match like terms

$$\left.\begin{array}{lll} x^2 & : & 4A = 16 \\ x & : & -8A = 0 \\ \text{cons.} & : & 2A = 0 \end{array}\right\} \Rightarrow A = 4 \text{ and } A = 0$$

$-8A = 0$ and $2A = 0$ $\Rightarrow A = 0$

(not possible!)

The process broke down because y_p was not the correct form.

Let's think of $g(x) = 16x^2$ as a 2^{nd} degree polynomial.

Assuming y_p is a 2^{nd} degree polynomial

Set $y_p = Ax^2 + Bx + C$ sub this

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Matching like terms

x^2: $4A = 16$

x: $-8A + 4B = 0$

const: $2A - 4B + 4C = 0$

Solve this system

Top eqn: $A=4$

middle eqn: $4B=8A \Rightarrow B=2A=8$

last eqn: $4C=-2A+4B \Rightarrow C=-\frac{1}{2}A+B=-2+8=6$

$A=4, \quad B=8, \quad C=6$

The particular solution is

$$\boxed{y_p=4x^2+8x+6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

That is, $A = -5$ and $A = 0$.

This is impossible as it would require $-5 = 0$. We failed to account for the cosine.

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Two Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.

Remark: The key point is that we have to account for **ALL** of the terms that might arise from taking derivatives.

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2^{nd} degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3^{rd} degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ (1[st] degree polynomial times e^{3x})

$$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ (two linear combos of sine/cosine)

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ (linear combo of 1^{st} poly times e^{-x} sine and 1^{st} poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$