October 4 Math 2306 sec. 54 Fall 2021

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

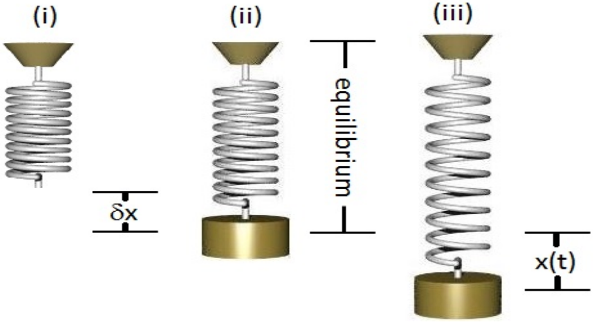

Figure: We track the displacement x from the equilibrium position. Recall δx is the displacement **in** equilibrium. We assume there are no damping nor external forces (for now).

Simple Harmonic Motion

We derive the equation for displacement. Given initial displacement x_0 and initial velocity x_1,

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1}$$

where $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion**[1].

k-spring constant, *m*-object's mass, *g*-acceleration due to gravity.

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[1] **Caution:** The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0\cos(\omega t) + \frac{x_1}{\omega}\sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[2]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[2]Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega}\sin(\omega t) = A\cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos\hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

No driving and no damping $\Rightarrow$

simple harmonic motion.

The ODE is $mx'' + kx = 0$,

in standard form

$$x'' + \omega^2 x = 0$$

we need to identify ω^2.

We don't have enough info to find

m and k, but we can use

$$\omega^2 = \frac{g}{\delta x}.$$

The displacement in equilibrium $\delta x = 6\text{in}$.

In US units $g = 32\ \text{ft}/\text{sec}^2$

$$\delta x = 6\text{in}\left(\frac{1\text{ft}}{12\text{in}}\right) = \frac{1}{2}\text{ft}$$

$$\omega^2 = \frac{32\ \text{ft}/\text{sec}^2}{\frac{1}{2}\text{ft}} = 64\ \frac{1}{\text{sec}^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

The ODE is $mx'' + kx = 0$, or in standard form $x'' + \omega^2 x = 0$.

Since $\delta x = 6\text{in}$, we know $\omega^2 = 64$.

Let's find m and k.

The mass $m = \frac{W}{g} = \frac{4\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{8}$ slug

The spring constant $k = \frac{W}{\delta x} = \frac{4\,lb}{\frac{1}{2}\,ft} = 8\,\frac{lb}{ft}$

The ODE is

$$\frac{1}{8}x'' + 8x = 0$$

$$\Longrightarrow \quad x'' + 64x = 0$$

We have initial conditions

$$x(0) = 4 \qquad x'(0) = -24$$

4 ft above equilibrium

24 ft/sec downward

The characteristic equation (in r) is

$$r^2 + 64 = 0$$

$$r^2 = -64$$

$$r = \pm i8 = 0 \pm i8$$

$$x_1 = e^{0t}\cos(8t) \quad , \quad x_2 = e^{0t}\sin(8t)$$

$$x(t) = c_1\cos(8t) + c_2\sin(8t)$$

Apply $x(0) = 4$, $x'(0) = -24$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \underbrace{\cos(0)}_{=1} + c_2 \underbrace{\sin(0)}_{=0} = 4$$

$$c_1 = 4$$

$$x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = -24$$

$$8c_2 = -24 \Rightarrow c_2 = -3$$

The equation of motion is

$$x = 4\cos(8t) - 3\sin(8t)$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

The frequency $f = \frac{1}{T} = \frac{4}{\pi}$

The amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5$$

The phase shift ϕ satisfies

$$\sin\phi = \frac{4}{5} \quad \text{and} \quad \cos\phi = \frac{-3}{5}.$$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$

about 127°

Free Damped Motion

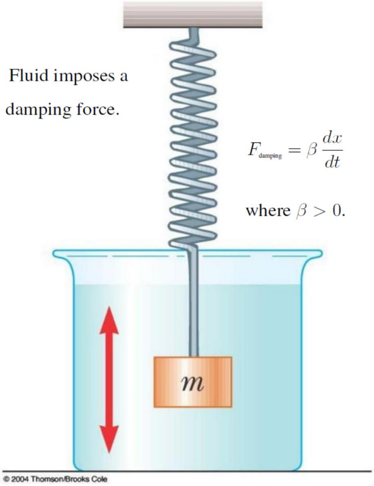


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

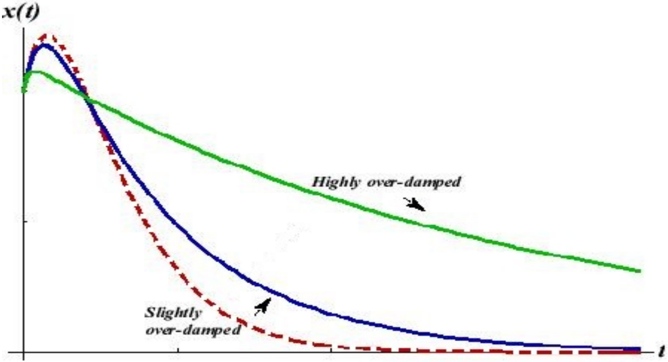


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

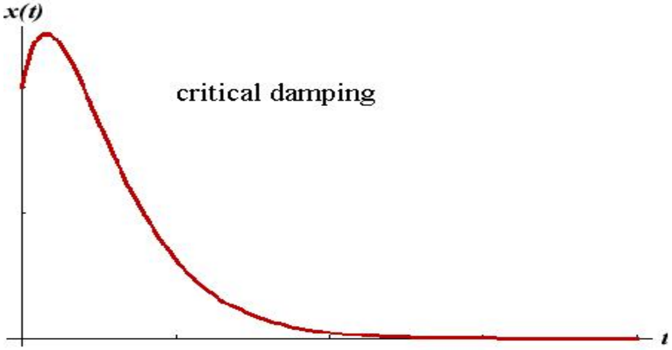


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

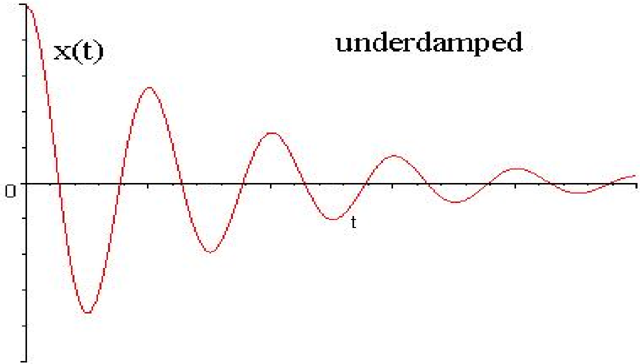
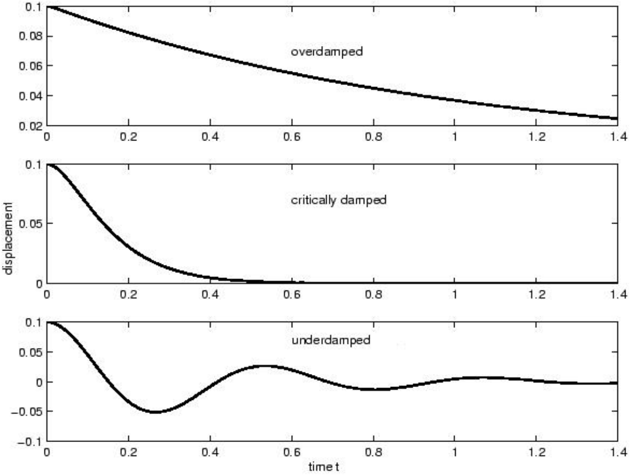


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping



2 real

1 real

complex conjugate

Figure: Comparison of motion for the three damping types.