October 5 Math 2306 sec. 51 Fall 2022

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these.

- The first equation has constant coefficient left side, but the tangent is not the right kind of right hand side.
- The second equation has an exponential right side, but the left side isn't constant coefficient.

We need another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

Recall $y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$y_p = u_1 y_1 + u_2 y_2 \qquad$ Find y_p'

$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$

$$\boxed{\text{Set} \quad u_1' y_1 + u_2' y_2 = 0}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect u_1, u_2, u_1', u_2'

$$\left(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{=0}\right)u_1 + \left(\underbrace{y_2'' + P(x)y_2' + Q(x)y_2}_{=0}\right)u_2 + y_1' u_1' + y_2' u_2' = g(x)$$

This gives $\boxed{y_1' u_1' + y_2' u_2' = g(x)}$

Now, we have two equations for u_1 and u_2.

$$y_1 u_1' + y_2 u_2' = 0$$

$$y_1' u_1' + y_2' u_2' = g(x)$$

Let's use Cramer's rule to solve.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}$$

Call the Wronskian W. Let

$$W_1 = \det \begin{bmatrix} 0 & y_2 \\ g & y_2' \end{bmatrix} = -g y_2$$

$$W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & g \end{bmatrix} = y_1 g$$

$$u_1' = \frac{W_1}{W} = \frac{-g y_2}{W} \Rightarrow u_1 = \int \frac{-g y_2}{W} \, dx$$

$$u_2' = \frac{W_2}{W} = \frac{g y_1}{W} \Rightarrow u_2 = \int \frac{g y_1}{W} \, dx$$

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p$$

where

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

We need y_p:

Standard form $\quad y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 8$

$$y_p = u_1y_1 + u_2y_2$$

$$y_1 = x^2 \qquad y_2 = x^{-2} \qquad g(x) = 8$$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2}) = -4x^{-1}$$

$$u_1 = \int \frac{-g\,y_2}{W}\,dx = \int \frac{-8x^{-2}}{-4x^{-1}}\,dx = 2\int \frac{x}{x^{-2}}\,dx$$

$$= 2\int \frac{1}{x}\,dx = 2\ln|x|$$

$$u_2 = \int \frac{g\,y_1}{W}\,dx = \int \frac{8x^2}{-4x^{-1}}\,dx = -2\int x^3\,dx$$

$$= \frac{-2}{4}x^4 = \frac{-1}{2}x^4$$

$y_1 = x^2,\quad y_2 = x^{-2}$

$$y_p = u_1 y_1 + u_2 y_2 =$$

$$= (2\ln x)\,x^2 + \left(\frac{-1}{2}x^4\right)x^{-2} = 2x^2\ln x - \frac{1}{2}x^2$$

The general solution

$$y = C_1 x^2 + C_2 x^{-2} + 2x^2 \ln x - \frac{1}{2} x^2$$

We can combine $C_1 x^2 - \frac{1}{2} x^2 = C_1 x^2$

$$y = C_1 x^2 + C_2 x^{-2} + 2x^2 \ln x$$

Apply the IC $y(1) = 0$ and $y'(1) = 0$

$$y' = 2C_1 x - 2C_2 x^{-3} + 4x \ln x + \frac{2x^2}{x}$$

$$y(1) = C_1 (1)^2 + C_2 (1)^{-2} + 2(1)^2 \ln 1 = 0$$

$$C_1 + C_2 = 0$$

$$y'(1) = 2c_1(1) - 2c_2(1)^{-3} + 4(1)\ln 1 + 2(1) = 0$$

$$2c_1 - 2c_2 + 2 = 0$$

$$2c_1 - 2c_2 = -2$$

Solve $\begin{aligned} c_1 + c_2 &= 0 \\ 2c_1 - 2c_2 &= -2 \end{aligned} \xrightarrow{\cdot 2} \begin{aligned} 2c_1 + 2c_2 &= 0 \\ 2c_1 - 2c_2 &= -2 \end{aligned}$

add $\quad 4c_1 = -2$

subtract $\quad 4c_2 = 2$

$$c_1 = \frac{-1}{2}, \quad c_2 = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{-1}{2}x^2 + \frac{1}{2}x^{-2} + 2x\ln x$$