# October 6 Math 3260 sec. 51 Fall 2025

## 3.8 Matrix Equations

**Matrix-Matrix Equation $AX = Y$**

The matrices $A$ and $Y$ are known. The variable to be solved for is the matrix $X$.

To solve such an equation, we set up the multiply-augmented matrix $\left[A \,\middle|\, Y\right]$ and do row reduction.

**Remark 1:** The matrix-matrix equation $AX = Y$ is equivalent to the whole collection of matrix-vector equations $A\,\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)$. This means we can interpret it as a **system of systems** of linear equations.

**Remark 2:** We are focusing on the square case, that is, where $A$, $Y$, and any solution $X$ would be $n \times n$ matrices.

**Theorem**

Suppose that $A$ and $Y$ are both $n \times n$ (square) matrices and consider the matrix equation $AX = Y$. Let

$$\widehat{A} = \left[ \begin{array}{c|c} A & Y \end{array} \right]$$

be the $n \times (2n)$ multiply-augmented matrix that corresponds to this matrix equation.

1. If $\text{rref}\,(A) = I_n$, then $AX = Y$ has a unique solution. Furthermore

   $$\text{rref}\left(\widehat{A}\right) = \left[ \begin{array}{c|c} I_n & X \end{array} \right]$$

   where $X$ is the unique solution of $AX = Y$.
2. If $\text{rref}\,(A) \neq I_n$, then $AX = Y$ is either inconsistent or has infinitely many solutions.

**Remark:** If any of the $n$ rightmost columns of $\widehat{A}$ is a pivot column, then $AX = Y$ is inconsistent.

# Recall Our Example

Solve the matrix equation $AX = Y$ or show that it is inconsistent.

$$\text{(a)}\quad A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

Last time, we set up the matrix $[A\,|\,Y] = \left[\begin{array}{cc|cc} -1 & -2 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right]$, and did row operations to get

$$\text{rref}\left([A\,|\,Y]\right) = \left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right].$$

So the equation $AX = Y$ is consistent and the solution $X = \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

Solve the two matrix-vector equations (with the column vectors from $Y$):

1. $A\vec{x}_1 = \langle -1, -2 \rangle$

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \vec{x}_1 = \langle -1, -2 \rangle$$

$$\left[\begin{array}{cc|c} -1 & -2 & -1 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$\vec{x}_1 = \langle 5, -2 \rangle$

2. $A\vec{x}_2 = \langle 1, 1 \rangle$

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \vec{x}_2 = \langle 1, 1 \rangle$$

$$\left[\begin{array}{cc|c} -1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$\vec{x}_2 = \langle -3, 1 \rangle$

# Observation

Solving $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$ is equivalent to solving the two vector equations

$$A\vec{x}_1 = \vec{y}_1 \quad \text{and} \quad A\vec{x}_2 = \vec{y}_2$$

where

$$\vec{x}_i = \mathrm{Col}_i(X) \quad \text{and} \quad \vec{y}_i = \mathrm{Col}_i(Y),$$

which is equivalent to solving two systems of linear equations

$$\begin{array}{rcrcr} -1x_{11} & - & 2x_{21} & = & -1 \\ 0x_{11} & + & 1x_{21} & = & -2 \end{array}, \quad \text{and} \quad \begin{array}{rcrcr} -1x_{12} & - & 2x_{22} & = & 1 \\ 0x_{12} & + & 1x_{22} & = & 1 \end{array}$$

# Recall Our Other Example

Prior to the exam, we solved the equation $AX = Y$, where

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad \text{and} \quad Y = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We found that

$$\left[A \,\middle|\, I_2\right] \quad \xrightarrow{rref} \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

This showed that $AX = I_2$ is consistent, and that the solution

$X = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

$AX = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The matrix $I_n$ is the **multiplicative identity** for both the matrix-vector and the matrix-matrix equation, $I_n\vec{x} = \vec{x}$ and $I_nA = A$ (for $n \times n$ matrix $A$). It has the same role for matrix multiplication as the number 1 has for scalar multiplication.

# 3.9 Matrix Inversion

Consider the rather straight forward problem:

$$\text{Solve for } x \text{ given} \quad 3x = 17.$$

To *solve for* or *isolate* $x$, we want the coefficient of $x$ to be 1, because 1 is the **identity**, i.e.,

$$1x = x$$

for multiplication of real numbers.

# 3.9 Matrix Inversion

If $a$ is a real number, we can ask whether there is another real number $b$ such that

$$ba = 1.$$

When such a number $b$ does exist, it is also the case that

$$ab = 1,$$

and we call $b$ the **multiplicative inverse** of $a$.

- The number zero does not have a multiplicative inverse because there is no number $b$ such that $b0 = 1$.
- If $a$ is **any nonzero** number, then there is a multiplicative inverse $a^{-1}$. (For real numbers, we also write a multiplicative inverse as $\frac{1}{a}$ and call it a reciprocal.)

# 3.9 Matrix Inversion

Consider the example we did earlier when we solved $AX = I_2$ for $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. We found

$$X = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Evaluate $XA$.

$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$XA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Definition**

Suppose that $A$ is an $n \times n$ matrix. We say that $A$ is **invertible** if there exists an $n \times n$ matrix $B$ such that $AB = I_n$ and $BA = I_n$.

If $B$ is a matrix such that $AB = I_n$ and $BA = I_n$, then we say that $B$ is an **inverse** of $A$.

Note that in our definition, we want the two products $AB$ and $BA$ to be the same, and to be the $n \times n$ identity matrix, if $B$ is an inverse of the matrix $A$.

# Example

$\underset{3\times 3}{A}\ \underset{3\times 3}{B}$ $\qquad$ $\underset{3\times 3}{B}\ \underset{3\times 3}{A}$

Show that $B$ is an inverse of $A$ where $B = \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and

$$A = \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}.$$

$$AB = \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Theorem: Inverses are Unique

Suppose $A$ is an invertible $n \times n$ matrix. Then the inverse of $A$ is unique.

This means that $A$ can't have two distinct inverses. That is, if $AB = BA = I_n$ and $AC = CA = I_n$, it must be that $B = C$.

**Proof:** Suppose $AB = BA = I_n$ and $AC = CA = I_n$.

Note that

$$B = BI_n = B(\underbrace{AC}_{I_n}) = (BA)C = I_nC = C$$

That is, $B = C$.

**Theorem:**

Suppose $A$ and $B$ are $n \times n$ matrices such that $AB = I_n$. Then $BA = I_n$.

**Remark:** This is not a trivial statement. It says that if we find a matrix $B$ such that $AB = I_n$, then we are guaranteed that $A$ and $B$ commute,

$$\text{i.e.,} \quad AB = BA.$$

And, in fact, that both

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

In order to prove this theorem, we need to remember an important previous theorem.

# Notation & The Invertibility Theorem

Don't ever write $\frac{1}{A}$.

If $B$ is the inverse of an invertible matrix, $A$, we will write

$$B = A^{-1}.$$

Now we answer two big questions: (1) Which matrices have inverses, and (2) how is an inverse computed?

**Invertibility Theorem**

Suppose that $A$ is an $n \times n$ matrix. Then $A$ is invertible if and only if $\mathrm{rref}(A) = I_n$. Moreover, if $A$ is invertible, and $\widehat{A}$ is the multiply-augmented matrix

$$\widehat{A} = \left[A \mid I_n\right],$$

then

$$rref\left(\widehat{A}\right) = \left[I_n \mid A^{-1}\right],$$

**Remark:** This gives a test for invertibility (the rref is or is not $I_n$), and a method for finding an inverse!

# Example

Determine whether $A$ is invertible. If invertible, find its inverse. (Let's do this by hand.)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Set up $[A \mid I_2]$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \quad 2R_1 + R_2 \to R_2$$

$$\begin{array}{cccc} -2 & 8 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 0 & 9 & 1 & 2 \end{array}\right] \quad \frac{1}{9}R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right] \qquad -4R_2 + R_1 \to R_1$$

$$\begin{array}{cccc} 0 & -4 & -\frac{4}{9} & -\frac{8}{9} \\ -1 & 4 & 0 & \frac{9}{9} \end{array}$$

$$\left[\begin{array}{cc|cc} -1 & 0 & -\frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right] \qquad -R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{9} & -\frac{1}{9} \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right]$$

$A^{-1}$ exists and $A^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Check:

$$\frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Example

Determine whether $A$ is invertible. If invertible, find its inverse.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -3/5 & 7/10 \\ 0 & 1 & -1 & 0 & 1/5 & -2/5 \\ 0 & 0 & 0 & 1 & 1/5 & 1/10 \end{array}\right]$$

$$\underbrace{\phantom{xxx}}_{\text{not } I_3}$$

A is not invertible.

# Example

Determine whether $A$ is invertible. If invertible, find its inverse.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 10 & -3 & -5 \\ -6 & 2 & 3 \end{bmatrix}$$

$$[A \mid I_3] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$A^{-1}$ exists and $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

## Matrix-Vector & Matrix-Matrix Equations

Suppose $A$ is an invertible $n \times n$ matrix. Then,

- for any vector $\vec{y}$ in $R^n$, the matrix-vector equation $A\vec{x} = \vec{y}$ has unique solution
$$\vec{x} = A^{-1}\vec{y},$$
- and for any $n \times p$ matrix $Y$, the matrix-matrix equation $AX = Y$ has unique solution
$$X = A^{-1}Y.$$

$AX = Y$ $\qquad$ $A^{-1}AX = A^{-1}Y$ $\qquad$ $X = A^{-1}Y$

$I_n X = A^{-1}Y$