October 6 Math 3260 sec. 53 Fall 2025

3.8 Matrix Equations

Matrix-Matrix Equation $AX = Y$

The matrices A and Y are known. The variable to be solved for is the matrix X.

To solve such an equation, we set up the multiply-augmented matrix $\left[A \,\middle|\, Y\right]$ and do row reduction.

Remark 1: The matrix-matrix equation $AX = Y$ is equivalent to the whole collection of matrix-vector equations $A\,\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)$. This means we can interpret it as a **system of systems** of linear equations.

Remark 2: We are focusing on the square case, that is, where A, Y, and any solution X would be $n \times n$ matrices.

Theorem

Suppose that A and Y are both $n \times n$ (square) matrices and consider the matrix equation $AX = Y$. Let

$$\widehat{A} = \left[\begin{array}{c|c} A & Y \end{array}\right]$$

be the $n \times (2n)$ multiply-augmented matrix that corresponds to this matrix equation.

1. If $\mathrm{rref}\,(A) = I_n$, then $AX = Y$ has a unique solution. Furthermore

$$\mathrm{rref}\left(\widehat{A}\right) = \left[\begin{array}{c|c} I_n & X \end{array}\right]$$

   where X is the unique solution of $AX = Y$.

2. If $\mathrm{rref}\,(A) \neq I_n$, then $AX = Y$ is either inconsistent or has infinitely many solutions.

Remark: If any of the n rightmost columns of $\widehat{A}$ is a pivot column, then $AX = Y$ is inconsistent.

Recall Our Example

Solve the matrix equation $AX = Y$ or show that it is inconsistent.

$$\text{(a)}\quad A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

Last time, we set up the matrix $[A\,|\,Y] = \left[\begin{array}{cc|cc} -1 & -2 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right]$, and did row operations to get

$$\text{rref}\left([A\,|\,Y]\right) = \left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right].$$

So the equation $AX = Y$ is consistent and the solution $X = \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

Solve the two matrix-vector equations (with the column vectors from Y):

1. $A\vec{x}_1 = \langle -1, -2 \rangle$

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \vec{x} = \langle -1, -2 \rangle$$

$$\left[\begin{array}{cc|c} -1 & -2 & -1 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$\vec{x}_1 = \langle 5, -2 \rangle$

2. $A\vec{x}_2 = \langle 1, 1 \rangle$

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \vec{x} = \langle 1, 1 \rangle$$

$\vec{x}_2 = \langle -3, 1 \rangle$

Observation

Solving $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$ is equivalent to solving the two vector equations

$$A\vec{x}_1 = \vec{y}_1 \quad \text{and} \quad A\vec{x}_2 = \vec{y}_2$$

where

$$\vec{x}_i = \mathrm{Col}_i(X) \quad \text{and} \quad \vec{y}_i = \mathrm{Col}_i(Y),$$

which is equivalent to solving two systems of linear equations

$$\begin{array}{rcrcr} -1x_{11} & - & 2x_{21} & = & -1 \\ 0x_{11} & + & 1x_{21} & = & -2 \end{array}, \quad \text{and} \quad \begin{array}{rcrcr} -1x_{12} & - & 2x_{22} & = & 1 \\ 0x_{12} & + & 1x_{22} & = & 1 \end{array}$$

Recall Our Other Example

Prior to the exam, we solved the equation $AX = Y$, where

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad \text{and} \quad Y = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We found that

$$\left[A \mid I_2\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

This showed that $AX = I_2$ is consistent, and that the solution $X = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

$$AX = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix I_n is the **multiplicative identity** for both the matrix-vector and the matrix-matrix equation, $I_n\vec{x} = \vec{x}$ and $I_nA = A$ (for $n \times n$ matrix A). It has the same role for matrix multiplication as the number 1 has for scalar multiplication.

3.9 Matrix Inversion

Consider the rather straight forward problem:

$$\text{Solve for } x \text{ given} \quad 3x = 17.$$

To *solve for* or *isolate* x, we want the coefficient of x to be 1, because 1 is the **identity**, i.e.,

$$1x = x$$

for multiplication of real numbers.

3.9 Matrix Inversion

If a is a real number, we can ask whether there is another real number b such that

$$ba = 1.$$

When such a number b does exist, it is also the case that

$$ab = 1,$$

and we call b the **multiplicative inverse** of a.

- The number zero does not have a multiplicative inverse because there is no number b such that $b0 = 1$.
- If a is **any nonzero** number, then there is a multiplicative inverse a^{-1}. (For real numbers, we also write a multiplicative inverse as $\frac{1}{a}$ and call it a reciprocal.)

3.9 Matrix Inversion

Consider the example we did earlier when we solved $AX = I_2$ for $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. We found

$$X = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Evaluate XA.

$$XA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$XA = AX$$

Definition

Suppose that A is an $n \times n$ matrix. We say that A is **invertible** if there exists an $n \times n$ matrix B such that $AB = I_n$ and $BA = I_n$.

If B is a matrix such that $AB = I_n$ and $BA = I_n$, then we say that B is an **inverse** of A.

Note that in our definition, we want the two products AB and BA to be the same, and to be the $n \times n$ identity matrix, if B is an inverse of the matrix A.

Example

$\underset{3\times 3}{A}\ \underset{3\times 3}{B}$ 3×3

$\underset{3\times 3}{B}\ \underset{3\times 3}{A}$ 3×3

Show that B is an inverse of A where $B = \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and

$A = \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 4 & 4 \\ 0 & -4 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -1 & -6 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Inverses are Unique

Suppose A is an invertible $n \times n$ matrix. Then the inverse of A is unique.

This means that A can't have two distinct inverses. That is, if $AB = BA = I_n$ and $AC = CA = I_n$, it must be that $B = C$.

Proof: Suppose $AB = BA = I_n$ and $AC = CA = I_n$.

Note that

$$B = BI_n = B(\underbrace{AC}_{I_n}) = (BA)C = I_nC = C.$$

That is, $B = C$.

Theorem:

Suppose A and B are $n \times n$ matrices such that $AB = I_n$. Then $BA = I_n$.

Remark: This is not a trivial statement. It says that if we find a matrix B such that $AB = I_n$, then we are guaranteed that A and B commute,

$$\text{i.e.,} \quad AB = BA.$$

And, in fact, that both

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

In order to prove this theorem, we need to remember an important previous theorem.

Notation & The Invertibility Theorem

If B is the inverse of an invertible matrix, A, we will write

$$B = A^{-1}.$$

Don't ever write $\frac{1}{A}$

Now we answer two big questions: (1) Which matrices have inverses, and (2) how is an inverse computed?

Invertibility Theorem

Suppose that A is an $n \times n$ matrix. Then A is invertible if and only if $\mathrm{rref}(A) = I_n$. Moreover, if A is invertible, and $\widehat{A}$ is the multiply-augmented matrix

$$\widehat{A} = [A \mid I_n],$$

then

$$rref\left(\widehat{A}\right) = [I_n \mid A^{-1}],$$

we're solving

$AX = I_n$

Remark: This gives a test for invertibility (the rref is or is not I_n), and a method for finding an inverse!

Example

Determine whether A is invertible. If invertible, find its inverse. (Let's do this by hand.)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Set up $[A \mid I_2]$ and row reduce

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \quad 2R_1 + R_2 \to R_2$$

$$\begin{array}{cccc} -2 & 8 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 0 & 9 & 1 & 2 \end{array}\right] \quad \frac{1}{9}R_2 \to R$$

$$\left[\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right] \quad -4R_2 + R_1 \to R_1$$

$$\begin{array}{cccc} 0 & -4 & -\frac{4}{9} & -\frac{8}{9} \\ -1 & 4 & 0 & \frac{9}{9} \end{array}$$

$$\left[\begin{array}{cc|cc} -1 & 0 & -\frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right] \quad -R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{9} & -\frac{1}{9} \\ 0 & 1 & \frac{1}{9} & \frac{2}{9} \end{array}\right]$$

$\mathrm{rref}(A) = I_2$ so A^{-1} exists and

$$A^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Check:

$$A^{-1}A = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Determine whether A is invertible. If invertible, find its inverse.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -3/5 & 7/10 \\ 0 & 1 & -1 & 0 & 1/5 & -2/5 \\ 0 & 0 & 0 & 1 & 1/5 & 1/10 \end{array}\right]$$

not I_3

A is not invertible

Example

Determine whether A is invertible. If invertible, find its inverse.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 10 & -3 & -5 \\ -6 & 2 & 3 \end{bmatrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ 10 & -3 & -5 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

A^{-1} exists and $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

Matrix-Vector & Matrix-Matrix Equations

Suppose A is an invertible $n \times n$ matrix. Then,

- for any vector $\vec{y}$ in R^n, the matrix-vector equation $A\vec{x} = \vec{y}$ has unique solution
$$\vec{x} = A^{-1}\vec{y},$$
- and for any $n \times p$ matrix Y, the matrix-matrix equation $AX = Y$ has unique solution
$$X = A^{-1}Y.$$

$AX = Y \qquad A^{-1}AX = A^{-1}Y \qquad I_n X = A^{-1}Y$

$X = A^{-1}Y$