October 7 Math 2306 sec. 53 Fall 2024

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{rx}, e^{2x}, $e^{-\pi x}$
- sines and/or cosines, $\sin(kx)$, $\cos(kx)$ k - constant
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left is constant coef., and the right side, $g(x) = 8x+1$, is a 1^{st} degree poly.

We'll suppose that y_p is also a 1^{st} degree polynomial. Set $y_p = Ax + B$ with A and B constants. Sub y_p into the ODE to see if we can find A and B.

[1] **We're only ignoring the y_c part to illustrate the process.**

$$y'' - 4y' + 4y = 8x + 1$$

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y_p'' - 4y_p' + 4y_p = 8x + 1$

$0 - 4(A) + 4(Ax + B) = 8x + 1$

Collect like terms

$\underline{4Ax} + (\underline{-4A + 4B}) = \underline{8x} + \underline{1}$

Match like terms:

$4A = 8 \Rightarrow A = 2$

$-4A + 4B = 1 \Rightarrow 4B = 1 + 4A = 1 + 4(2) = 9$

$\Rightarrow B = \frac{9}{4}$

$$\boxed{\text{we found a particular solution } y_p = 2x + \frac{9}{4}}$$

$$y'' - 4y' + 4y = 8x + 1$$

Check:

$y_p = 2x + \frac{9}{4}$

$y_p' = 2$

$y_p'' = 0$

$y_p'' - 4y_p' + 4y_p =$

$0 - 4(2) + 4(2x + \frac{9}{4}) =$

$-8 + 8x + 9 =$

$8x + 1$ ✓

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

The left side is constant coef, and the right side $g(x) = 6e^{-3x}$ is an exponential.

Set $y_p = Ae^{-3x}$.

We could set $y_p = Ae^{Bx}$. We'd find $B = -3$.

Sub y_p into the ODE.

$y_p = Ae^{-3x}$

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$$y'' - 4y' + 4y = 6e^{-3x}$$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$9Ae^{-3x} + 12Ae^{-3x} + 4Ae^{-3x} = 6e^{-3x}$$

$$e^{-3x}(9A + 12A + 4A) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

$$\Rightarrow 25A = 6$$

$$A = \frac{6}{25}$$

we found particular solution

$$\boxed{y_p = \frac{6}{25}e^{-3x}}$$

The Initial Guess Must Be General in Form

Find a particular solution to $y'' - 4y' + 4y = 16x^2$

The left side is constant coef, and $g(x) = 16x^2$ is the correct type of right side. We could classify $g(x)$ as a monomial or as a polynomial.

Let's set $y_p = Ax^2$. Sub this in

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$2A - 8Ax + 4Ax^2 = 16x^2$$

Match like terms

$$4A = 16 \Rightarrow A = 4$$

$$\left.\begin{aligned} -8A &= 0 \\ 2A &= 0 \end{aligned}\right\} \Rightarrow A = 0$$

impossible!

Our set up for y_p is incorrect.

$$y'' - 4y' + 4y = 16x^2$$

We need to consider $g(x) = 16x^2$ as a 2nd degree polynomial. Assume y_p is also a 2nd degree poly.

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

Collect like terms:

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Match like terms

$$\begin{aligned} 4A &= 16 \\ -8A + 4B &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}$$

$4A = 16 \Rightarrow A = 4$

$4B = 8A \Rightarrow B = 2A = 2(4) = 8$

$4C = -2A + 4B = -2(4) + 4(8) = 24$

$C = 6$

$$\boxed{\text{we found particular solution} \quad y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Two Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2nd degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3rd degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = 12e^{-4x}$ (constant multiple of e^{-4x})

$$y_p = Ae^{-4x}$$

(f) $g(x) = xe^{3x}$ (1^{st} degree polynomial times e^{3x})

$$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

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The correct form is a whole quadratic times a
sine plus another whole quadratic times a
cosine. There are six coefficients for the
six types of like terms that can arise because
of differentiation.
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