October 8 Math 3260 sec. 53 Fall 2025

3.9 Matrix Inversion

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

Key Results

- If A and B are $n \times n$ and $AB = I_n$, then $BA = I_n$.
- If A is invertible, then it has exactly one inverse.
- We write the inverse of a matrix A as A^{-1}.

Main Invertibility Test & Algorithm

Invertibility Theorem

Suppose that A is an $n \times n$ matrix. Then A is invertible if and only if $\text{rref}(A) = I_n$. Moreover, if A is invertible, and $\widehat{A} = [A \mid I_n]$, is the multiply-augmented matrix then

$$rref\left(\widehat{A}\right) = [I_n \mid A^{-1}],$$

This gives a test for invertibility:

$$\begin{array}{rcccl} \text{rref}(A) & = & I_n & \Longrightarrow & A \text{ is invertible} \\ \text{rref}(A) & \neq & I_n & \Longrightarrow & A \text{ is not invertible.} \end{array}$$

The algorithm for finding A^{-1} is $\text{rref}\left([A \mid I_n]\right) = [I_n \mid A^{-1}]$.

Matrix-Vector & Matrix-Matrix Equations

Suppose A is an invertible $n \times n$ matrix. Then,

- for any vector $\vec{y}$ in R^n, the matrix-vector equation $A\vec{x} = \vec{y}$ has unique solution
$$\vec{x} = A^{-1}\vec{y},$$
- and for any $n \times p$ matrix Y, the matrix-matrix equation $AX = Y$ has unique solution
$$X = A^{-1}Y.$$

Example

Use a matrix inverse[1] to solve the system.

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 3 \\ -x_1 & + & 4x_2 & = & 2 \end{array}$$

Formulate as $A\vec{x} = \vec{y}$

$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle 3, 2 \rangle$

$A^{-1} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

[1] Last time, we found that

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}^{-1} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}.$$

$$\vec{x} = A^{-1}\vec{y} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\langle 3, 2\rangle = \frac{1}{9}\langle 4(3)+(-1)(2), 1(3)+2(2)\rangle$$

$$= \frac{1}{9}\langle 10, 7\rangle = \left\langle \frac{10}{9}, \frac{7}{9}\right\rangle$$

So $x_1 = \frac{10}{9}$, $x_2 = \frac{7}{9}$.

$$\begin{aligned} 2x_1 &+ x_2 &= 3 \\ -x_1 &+ 4x_2 &= 2 \end{aligned}$$

Check: $2\left(\frac{10}{9}\right) + \frac{7}{9} = \frac{20+7}{9} = \frac{27}{9} = 3$

$-\left(\frac{10}{9}\right) + 4\left(\frac{7}{9}\right) = \frac{-10+28}{9} = \frac{18}{9} = 2$ ✓

Products & Transposes

Theorem

If A and B are invertible $n \times n$ matrices, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Theorem

If A is an invertible $n \times n$ matrix, then A^T is invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

Proof that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$: Recall $(XY)^T = Y^T X^T$

$(A^T)^{-1}$ is **the** matrix having the property that $(A^T)^{-1}A^T = A^T(A^T)^{-1} = I_n$.

Note that $AA^{-1} = I_n$ Take the transpose

$(AA^{-1})^T = I_n^T = I_n$

$(A^{-1})^T A^T = I_n$

↑ this is the inverse of A^T

Hence $(A^{-1})^T = (A^T)^{-1}$.

Chapter 4 Vector Spaces & Subspaces

In this chapter, we will

- learn about additional properties of vectors in R^n,
- learn about special subsets of R^n, including some related to matrices,
- state the **Fundamental Theorem of Linear Algebra**,
- and pin down precisely what a **vector space** is.

4.1 Linear Independence

Let's recall that the matrix-vector product

$$A\vec{x} = x_1 \operatorname{Col}_1(A) + x_2 \operatorname{Col}_2(A) + \cdots + x_n \operatorname{Col}_n(A).$$

$A\vec{x} = \vec{y}$

The matrix-vector equation $A\vec{x} = \vec{y}$ is consistent if and only if

$$\vec{y} \in \operatorname{Span}\left\{\operatorname{Col}_1(A), \operatorname{Col}_2(A), \ldots, \operatorname{Col}_n(A)\right\}.$$

Let's recall that the homogeneous equation

$$A\vec{x} = \vec{0}_m.$$

is always consistent. The zero vector is always in a span.

$$x_1 \operatorname{Col}_1(A) + x_2 \operatorname{Col}_2(A) + \cdots + x_n \operatorname{Col}_n(A) = \vec{0}_m$$

There are two possibilities:

1. The equation has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$, or

2. the equation has nontrivial solution—i.e. there are numbers x_1, x_2, etc. that are **not all zero**, that make the equation true.

We can think of these two posibilities as some sort of **property** of the set of vectors

$$\{\operatorname{Col}_1(A), \operatorname{Col}_2(A), \ldots, \operatorname{Col}_n(A)\}.$$

Example

Consider the vectors in R^3

$$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle.$$

Find a matrix A such that

$$\text{Span}\left\{\text{Col}_1(A), \text{Col}_2(A), \dots, \text{Col}_n(A)\right\} = \text{Span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}.$$

An obvious choice is to set

$\text{Col}_1(A) = \vec{v}_1$, $\text{Col}_2(A) = \vec{v}_2$ and $\text{Col}_3(A) = \vec{v}_3$.

This makes

$$A = \begin{bmatrix} -2 & -5 & 3 \\ 4 & 8 & 0 \\ -5 & -6 & -12 \end{bmatrix}$$

Linear Independence/Dependence

Given any set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^m, we can consider the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m.$$

There are always two possible cases,

1. The equation has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$, or
2. the equation has nontrivial solution—i.e. there are numbers x_1, x_2, etc. that are **not all zero**, that make the equation true.

Which of these is true is a property of the set of vectors, $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$.

Definition: Linear Independence

The collection of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^m is said to be **linearly independent** if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m \tag{1}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$.

If the collection of vectors is not linearly independent, then we say that it is **linearly dependent.**

For a linearly dependent set of vectors, an equation of the form (1) having at least one nonzero weight is called a **linear dependence relation.**

Linear Dependence Relation Example

Consider the collection of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \langle 1, 0, 1, 0 \rangle, \quad \vec{v}_2 = \langle 0, 1, 0, 1 \rangle, \quad \text{and} \quad \vec{v}_3 = \langle 2, -1, 2, -1 \rangle.$$

Note that $\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$. State a **linear dependence relation** for the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

We want an equation of the form

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}_4$ for some numbers x_1, x_2, x_3 not all zero. Rearrange the given equation to get

$$2\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}_4$$

This is a linear dependence relation w/ $x_1 = 2$, $x_2 = -1$ and $x_3 = -1$.

Example: Set of One Vector

What can we say about a set of only one vector?

1. Is the set $\{\langle 1, 0, 2, 1\rangle\}$ linearly dependent or linearly independent?
2. Is the set $\{\langle 0, 0, 0\rangle\}$ linearly dependent or linearly independent?

1. Consider the equation

$$x_1 \langle 1, 0, 2, 1\rangle = \langle 0, 0, 0, 0\rangle$$

$$\langle x_1, 0, 2x_1, x_1\rangle = \langle 0, 0, 0, 0\rangle$$

This is only true if $x_1 = 0$.

So $\{\langle 1, 0, 2, 1\rangle\}$ is linearly independent.

2. Consider $x_1 \langle 0,0,0\rangle = \langle 0,0,0\rangle$

This is true for any real number x_1.

In fact

$$1\langle 0,0,0\rangle = \langle 0,0,0\rangle$$

is a linear dependence relation.

So $\{\langle 0,0,0\rangle\}$ is linearly dependent.

Turns out, a set of one vector in $\mathbb{R}^m$ is linearly independent if that vector is not the zero vector and is linearly dependent if the vector is the zero vector in $\mathbb{R}^m$.