October 9 Math 2306 sec. 51 Fall 2024

Section 9: Method of Undetermined Coefficients

We are considering nonhomogeneous, linear ODEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

with restrictions on the left and right sides.

- The left side must be **constant coefficient.**
- The right side, $g(x)$, has to be one of the following function types:
  - polynomials,
  - exponentials,
  - sines and/or cosines,
  - and products and sums of the above kinds of functions

The **general solution** will have the form $y = y_c + y_p$. The process here is for finding y_p.

The Method of Undetermined Coefficients

We saw some examples last time. The basic process is

- Confirm the ODE has the right properties and classify the function g on the right.
- Set up an ansatz[1] for y_p by assuming it is the same *type* of function as g but with unknown coefficients.
- Substitute the assumed y_p into the ODE and match *like terms* to find the coefficients that work.

Remark: The complementary solution is found using the process in the last section. This will be a critical part of the process and will usually be done **first**.

[1] An **ansatz** is a solution *guess*. It's generally a well informed, educated guess based on the type of problem under consideration. An ansatz typically includes some unspecified features that are to be found in the problem solving process

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$$y_p = A$$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$$y_p = Ax + B$$

(c) $g(x) = 5x^2$ (2^{nd} degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3^{rd} degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = 12e^{-4x}$ (constant multiple of e^{-4x})

$$y_p = Ae^{-4x}$$

(f) $g(x) = xe^{3x}$ (1^{st} degree polynomial times e^{3x})

$$y_p = (Ax + B)e^{3x}$$

Remark: The last example can also be written as $y_p = Axe^{3x} + Be^{3x}$. The key point is that the factor x in g needs to be thought of as a first degree polynomial.

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Remark: Note that there are exactly six like terms, $x^2\sin(3x)$, $x^2\cos(3x)$, $x\sin(3x)$, $x\cos(3x)$, $\sin(3x)$, and $\cos(3x)$. They each need their own coefficient, $A, B, \ldots, F$.

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ (linear combo of 1^{st} poly times e^{-x} sine and 1^{st} poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.
- Constants inside of sines, cosines, and exponentials (e.g., the "2" in e^{2x} or the "π" in $\sin(\pi x)$) are not undetermined. We don't change those.

Caution

- The method is self correcting, meaning if the initial *guess* is wrong, it will become apparant. But it's best to get the set up correct to avoid unnecessary work.
- *Constant* really means **constant**. None of the coefficients can end up depending on the variable x.
- The form of y_p can depend on y_c, but this hasn't been considered yet. (We'll come back to this shortly.)

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

we can consider two sub problems

$$y'' - 4y' + 4y = 6e^{-3x} \quad \text{and}$$

$$y'' - 4y' + 4y = 16x^2$$

For $g_1(x) = 6e^{-3x}$, $y_{p_1} = Ae^{-3x}$

$g_2(x) = 16x^2$, $y_{p_2} = Bx^2 + Cx + D$

For the whole ODE,

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

What happens if the assumed form for y_p is part[2] of y_c? Consider applying the process to find a particular solution to the ODE

$$y'' - 2y' = 3e^{2x}$$

$g(x) = 3e^{2x}$, set $y_p = Ae^{2x}$

sub into the ODE.

$y_p = Ae^{2x}$, $y_p' = 2Ae^{2x}$, $y_p'' = 4Ae^{2x}$

$$y_p'' - 2y_p' = 3e^{2x}$$
$$4Ae^{2x} - 2(2Ae^{2x}) = 3e^{2x}$$
$$0 = 3e^{2x}$$

This is false for all values of A.

[2] A term in $g(x)$ is contained in a fundamental solution set of the associated homogeneous equation.

Our choice for $y_p = Ae^{2x}$ is a solution to the associated homogeneous equation.

Find y_c: $y_c'' - 2y_c' = 0$

Characteristic eqn $r^2 - 2r = 0 \Rightarrow r(r-2) = 0$

$r = 0, \; r = 2$

$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}$

We can try to salvage the method by multiplying our guess y_p by a factor of x. Set $y_p = (Ae^{2x})x = Axe^{2x}$

$$y'' - 2y' = 3e^{2x}$$

Sub our guess into the ODE.

$$y_p = A x \cdot e^{2x}$$

$$y_p' = A e^{2x} + 2Axe^{2x}$$

$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

$$y_p'' - 2y_p' = 3e^{2x}$$

$$4Ae^{2x} + 4Axe^{2x} - 2\left(Ae^{2x} + 2Axe^{2x}\right) = 3e^{2x}$$

Collect like terms e^{2x}, xe^{2x}

$$xe^{2x}(4A-4A) + e^{2x}(4A-2A) = 3e^{2x}$$

$$2Ae^{2x} = 3e^{2x}$$

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

The particular solution is

$$y_p = \frac{3}{2} x e^{2x}$$

Using $y_1 = 1$, $y_2 = e^{2x}$, the general solution is

$$y = c_1 + c_2 e^{2x} + \frac{3}{2} x e^{2x}$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_i(x)$$

The first thing we do is solve the associated homogeneous equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0,$$

for the complementary solution y_c.

Case 1:

We write out our guess for y_{p_i} using the general rules of thumb and principles already discussed (see all the examples we went through). We compare our guess for y_{p_i} to y_c and **there are no like terms in common**.

We have the correct form for y_{p_i} so we start the substitution process and complete finding our particular solution.

Remark: All the examples so far, up to the slide that says "A Glitch!," were Case 1 examples.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_i(x)$$

Case 2:

We write out our guess for y_{p_i} using the general rules of thumb and principles already discussed. We compare our guess for y_{p_i} to y_c and **there is one or more like terms in common between y_{p_i} and y_c.**

We have to adjust our form of y_{p_i}. We do this by multiplying the whole function y_{p_i} by a factor of x^n, where n is the smallest positive integer such that our new y_{p_i} has no like terms in common with y_c.

Once we have the correct format for y_{p_i}, we start the substitution process and complete finding our particular solution.

Remark: In practice, we can multiply by x. If the new y_{p_i} still has a like term in common with y_c, multiply by x again. Continue to multiply by x until there are no common like terms left. That is, we don't have to know what n is up front.