October 9 Math 2306 sec. 51 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

The general solution, $y = y_c + y_p$ will require both y_c and y_p. The associated homogeneous equation will be constant coefficient, so we use the method of the last section to find y_c.

Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

- Classify g as a certain *type*, and assume y_p is of this same type[1] with unspecified coefficients, A, B, C, etc.
- Substitute the $\underbrace{\text{assumed } y_p}_{\text{ansatz}}$ into the ODE and collect like terms
- Match like terms on the left and right to get equations for the coefficients.
- Solve the resulting system to determine the coefficients for y_p.

[1] We will see shortly that our final conclusion on the format of y_p can depend on y_c.

Some Rules & Caveats

Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.
- Constants inside of sines, cosines, and exponentials (e.g., the "2" in e^{2x} or the "π" in $\sin(\pi x)$) are not undetermined. We don't change those.

Caution

- The method is self correcting, but it's best to get the set up correct.
- The form of y_p can depend on y_c. (More about this later.)

Examples of Forms of y_p based on g (Trial Guesses)

We were considering different types of right hand side functions to determine the correct format for y_p.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = Ae^x\cos(2x) + Be^x\sin(2x)$$

(j) $g(x) = xe^{-x}\sin(\pi x)$ (linear combo of 1^{st} poly times e^{-x} sine and 1^{st} poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x}\sin(\pi x) + (Cx+D)e^{-x}\cos(\pi x)$$

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

Right Hand Side	Particular Solution Set Up
$g(x) = 1$	$y_p = A$
$g(x) = x - 7$	$y_p = Ax + B$
$g(x) = 5x^2$	$y_p = Ax^2 + Bx + C$
$g(x) = 3x^3 - 5$	$y_p = Ax^3 + Bx^2 + Cx + D$
$g(x) = xe^{3x}$	$y_p = (Ax + B)e^{3x}$
$g(x) = \cos(7x)$	$y_p = A\cos(7x) + B\sin(7x)$
$g(x) = \sin(2x) - \cos(4x)$	$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$
$g(x) = x^2\sin(3x)$	$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$
$g(x) = e^x\cos(2x)$	$y_p = Ae^x\cos(2x) + Be^x\sin(2x)$
$g(x) = xe^{-x}\sin(\pi x)$	$y_p = (Ax + B)e^{-x}\sin(\pi x) + (Cx + D)e^{-x}\cos(\pi x)$

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

We can find y_{p_1} that solves

$$y_{p_1}'' - 4y_{p_1}' + 4y_{p_1} = 6e^{-3x} \qquad g_1(x) = 6e^{-3x}$$

$$y_{p_1} = Ae^{-3x}$$

Then find y_{p_2} that solves

$$y_{p_2}'' - 4y_{p_2}' + 4y_{p_2} = 16x^2 \qquad g_2(x) = 16x^2$$

$$y_{p_2} = Bx^2 + Cx + D$$

Then $y_p = y_{p_1} + y_{p_2}$

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

What happens if the assumed form for y_p is part[2] of y_c? Consider applying the process to find a particular solution to the ODE

$$y'' - y' = 3e^x$$

Constant coef left, exponential right.

$g(x) = 3e^x$ constant times e^x

Put $y_p = Ae^x$ substitute

$y_p' = Ae^x$

$y_p'' = Ae^x$

$y_p'' - y_p' = 3e^x$

$Ae^x - Ae^x = 3e^x$

$0 = 3e^x$

this is false for all possible values of A.

Our assumed y_p is part of y_c.

[2]A term in $g(x)$ is contained in a fundamental solution set of the associated homogeneous equation.

Let's consider y_c: $y_c'' - y_c' = 0$

Characteristic eqn $m^2 - m = 0 \Rightarrow m(m-1) = 0$ real
$m = 0$ or $m = 1$ two roots

$y_1 = e^{0x} = 1$, $y_2 = e^{1x} = e^x$

$y_c = c_1 + c_2 e^x$

Based on experience w/ homogeneous equations,
we can try multiplying our y_p by x. i.e.,

try $y_p = x(Ae^x) = Axe^x$ sub this

$y_p' = Axe^x + Ae^x$

$y_p'' = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$

$y_p'' - y_p' = 3e^x$

$Axe^x + 2Ae^x - (Axe^x + Ae^x) = 3e^x$

Collect xe^x and e^x terms

$xe^x(A-A) + e^x(2A-A) = 3e^x$

$Ae^x = 3e^x \Rightarrow A=3$

we found particular solution

$y_p = 3xe^x$

The general solution, $y_c + y_p$, is

$y = c_1 + c_2 e^x + 3xe^x$

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Case II Examples

Find the general solution of the ODE.

$$y'' - 2y' + y = -4e^{x}$$

The left side is constant coef and the right is an exponential. Let's find y_c.

Characteristic eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ double root

$y_1 = e^{x}$ $\quad y_2 = xe^{x} \Rightarrow \boxed{y_c = c_1 e^{x} + c_2 x e^{x}}$

Now find y_p: $g(x) = -4e^{x}$ constant times e^{x}

$y_p = Ae^{x}$ like term w/ y_1

correct $\quad y_p = (Ae^{x})x = Axe^{x}$ like term w/ y_2

$y_p = x(Axe^x) = Ax^2e^x$ This is the correct form.

Sub:

$$y_p = Ax^2e^x$$
$$y_p' = Ax^2e^x + 2Axe^x$$
$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$
$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect e^x, xe^x, x^2e^x

$$x^2e^x(A - 2A + A) + xe^x(4A - 4A) + 2Ae^x = -4e^x$$

$$2Ae^x = -4e^x$$

$$A = -2$$

Hence $y_p = -2x^2e^x$

The general solution is

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ Double root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

Look for y_{p_1} that solves

$y_{p_1}'' - 4y_{p_1}' + 4y_{p_1} = \sin(4x)$ $g_1(x) = \sin(4x)$

$y_{p_1} = A\sin(4x) + B\cos(4x)$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

Find y_{p_2} that solves

$$y_{p_2}'' - 4y_{p_2}' + 4y_{p_2} = xe^{2x} \qquad g_2(x) = xe^{2x}$$

$$y_{p_2} = (Cx+D)e^{2x} = Cxe^{2x} + De^{2x}$$

$$y_{p_2} = (Cx+D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x}$$

$$y_{p_2} = (Cx+D)x^2e^{2x} = Cx^3e^{2x} + Dx^2e^{2x}$$

$$\boxed{y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}}$$