Partial Fraction Basics

We wish to decompose a **proper** rational function. To be proper means that the numerator has a degree that is stricly smaller than the denominator degree.

Factor types: There is a theorem in Algebra that says that every polynomial with real coefficients can be factored as a product of linear and irreducible quadratic factors[1]. So you can only have two kinds of factors:

1. **Linear** factors such as $(s+1)$, or $(s-3)$, or in general $(s-a)$, and
2. **Irreducible**[2] **Quadratic** factors such[3] as (s^2+4), or (s^2+2s+2), or more generally (s^2+bs+c) with $b^2-4c<0$.

Repeating vs Nonrepeating: A factor is referred to as **repeating** if it appears more than once in the factored polynomial. In other words, it's repeating if it has a power greater than 1 on it. For example, consider

$$\frac{s^2+3s+3}{(s-1)(s+2)^3(s^2+1)(s^2+2s+3)^5}$$

There are four factors in the denominator. We can characterize each one.

- $(s-1)$ is linear and it is not repeating (a.k.a. nonrepeating),
- $(s+2)$ is linear and it is repeating (since it's raised to the power 3),
- (s^2+1) is irreducible quadratic and it is nonrepeating,
- (s^2+2s+3) is irreducible quadratic and it is repeating (due to the power of 5).

Proper Set Up of Decomposition: There is a general rule for how to set up the terms in your decomp based on which type of factor you have.

1. **Linear** factors get a contant numerator. For example $\dfrac{A}{s+1}$ or $\dfrac{B}{s-3}$
2. **Irreducible Quadratic** factors get a line as the numerator. For example $\dfrac{As+B}{s^2+4}$ or $\dfrac{Cs+D}{s^2+2s+2}$

[1] Knowing it can be done, and actually being able to do it, are not the same thing. Factoring might be quite difficult in practice.

[2] Irreducible means that it can't be factored as a product of linear factors without introducing complex numbers.

[3] More generally, a linear factor can look like $(as+b)$, and a quadratic can look like (as^2+bs+c).

3. **Repeated Linear:** The decomposition will have as many terms as the power on the repeated factor. There is one term for each power. All terms have a constant numerator. For example if $(s+2)^3$ is a repeated factor, the decomposition will include
$$\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{(s+2)^3}$$
4. **Repeated Quadratic:** The same principle holds as with repeated linear except that each numberator is a line. For example, if $(s^2+2s+3)^5$ is a factor, then the decomposition will have five terms
$$\frac{As+B}{s^2+2s+3}+\frac{Cs+D}{(s^2+2s+3)^2}+\frac{Es+F}{(s^2+2s+3)^3}+\frac{Gs+H}{(s^2+2s+3)^4}+\frac{Js+K}{(s^2+2s+3)^5}$$

Example: Here is an example of the correct format for a partial fraction decomposition.

$$\frac{s^2+3s+3}{(s-1)(s+2)^3(s^2+1)(s^2+2s+3)^5}=$$
$$\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3}+\frac{Es+F}{s^2+1}+\frac{Gs+H}{s^2+2s+3}+$$
$$\frac{Js+K}{(s^2+2s+3)^2}+\frac{Ls+M}{(s^2+2s+3)^3}+\frac{Ns+P}{(s^2+2s+3)^4}+\frac{Qs+R}{(s^2+2s+3)^5}$$

Exercises: Determine what the partial fraction decomposition[4] should look like (i.e. write out the format). Some of these will require factoring first. Answers on the next page.

1. $\dfrac{s+2}{(s-1)(s+3)(s-4)}$
2. $\dfrac{6}{(s^2+5)(s-1)^3}$
3. $\dfrac{s^2-2}{s^3(s^2+s+1)^2}$
4. $\dfrac{4}{s^2-3s-4}$
5. $\dfrac{s+3}{(s^2-1)^2(s^2+4)^2}$
6. $\dfrac{2}{s^3-s^2}$
7. $\dfrac{s^2+1}{s^2-1}$

[4]One of these is a trick question.

Exercise Solutions:

1. $\dfrac{s+2}{(s-1)(s+3)(s-4)} = \dfrac{A}{s+2} + \dfrac{B}{s-1} + \dfrac{C}{s-4}$

2. $\dfrac{6}{(s^2+5)(s-1)^3} = \dfrac{As+B}{s^2+5} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} + \dfrac{E}{(s-1)^3}$

3. $\dfrac{s^2-2}{s^3(s^2+s+1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{Ds+E}{s^2+s+1} + \dfrac{Fs+G}{(s^2+s+1)^2}$

4. $\dfrac{4}{s^2-3s-4} = \dfrac{A}{s+1} + \dfrac{B}{s-4}$

5. $\dfrac{s+3}{(s^2-1)^2(s^2+4)^2} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+1} + \dfrac{D}{(s+1)^2} + \dfrac{Es+F}{s^2+4} + \dfrac{Gs+H}{(s^2+4)^2}$

6. $\dfrac{2}{s^3-s^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1}$

7. $\dfrac{s^2+1}{s^2-1}$ This is a trick question! The rational function is NOT proper. We can write it as

$$\frac{s^2+1}{s^2-1} = 1 + \frac{2}{s^2-1} = 1 + \frac{A}{s-1} + \frac{B}{s+1}.$$

Notes:

4. $s^2-3s-4 = (s+1)(s-4)$

5. $(s^2-1)^2(s^2+4)^2 = [(s-1)(s+1)]^2(s^2+4)^2 = (s-1)^2(s+1)^2(s^2+4)^2$

6. $s^3 - s^2 = s^2(s-1)$

7. $s^2+1 = (s^2-1)+2 \implies \dfrac{s^2+1}{s^2-1} = \dfrac{s^2-1+2}{s^2-1} = \dfrac{s^2-1}{s^2-1} + \dfrac{2}{s^2-1} = 1 + \dfrac{2}{s^2-1} = 1 + \dfrac{2}{(s-1)(s+1)}$

This can also be accomplished by performing long division

$$s^2-1\,\overline{)\,s^2+1}$$