MATH 1190 Calculus I

Section 4.2

Overview

- Extreme Values
- Absolute Extrema
- Extreme Value Theorem (EVT)
- Local Extrema
- Critical Values
- Closed Interval Method

Announcements & Questions

Extreme Values

- One of the most important applications of the derivative is its use as a tool to find the *minimum or maximum* values of a function.
- We can find **extreme values** (or **extrema**) of f over the entire domain (global) or over a particular interval I (local).
- There can be a unique optimal value, more than one, or none at all.

*Note on language:
The singular forms are minimum, maximum, and extremum.
The plural forms are minima, maxima, and extrema.

KENNESAW STATE UNIVERSITY

Absolute Extrema

Let f be a function on an interval I and let $a \in I$. We say that $f(a)$ is the

- **absolute minimum** of f on I if $f(a) \leq f(x)$ for all $x \in I$.
- **absolute maximum** of f on I if $f(a) \geq f(x)$ for all $x \in I$.

The extreme values *occur at* the x-values, but the extrema are the corresponding $\boldsymbol{y}$**-values**.

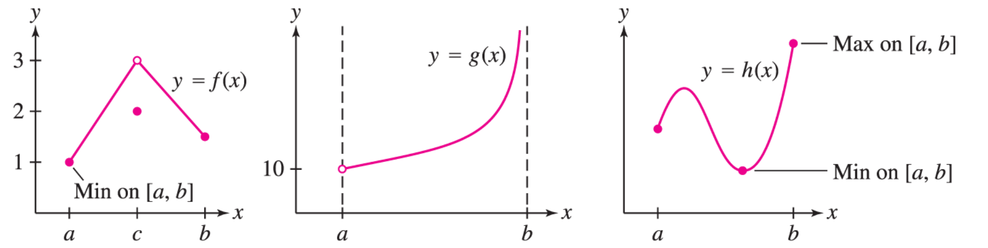
- Does every function have absolute extrema? Consider $y = x$ or $y = \frac{1}{x}$.

Extreme Value Theorem (EVT)

Theorem: Existence of Extrema on a Closed Interval
A continuous function f on a closed (bounded) interval $I = [a, b]$ takes on both an absolute maximum M and absolute minimum m on I.

What can go wrong?

- Discontinuity (A)
- Open Interval (B)



(A) Discontinuous function with no max on $[a, b]$, and a min at $x = a$.

(B) Continuous function with no min or max on the open interval (a, b).

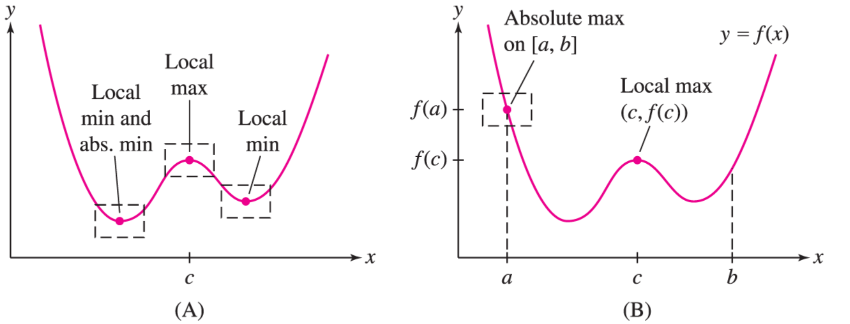
(C) Every continuous function on a closed interval $[a, b]$ has both a min and a max on $[a, b]$.

Local Extrema

We say that $f(c)$ is a

- **local minimum** occurring at $x = c$ if $f(c)$ is the minimum value of f on some open interval containing c.
- **local maximum** occurring at $x = c$ if $f(c)$ is the maximum value of f on some open interval containing c.

That is, $f(c)$ is a local extrema if it is greater (or smaller) than all other *nearby* values.



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

Example 0

How to identify and classify all extreme values of f.

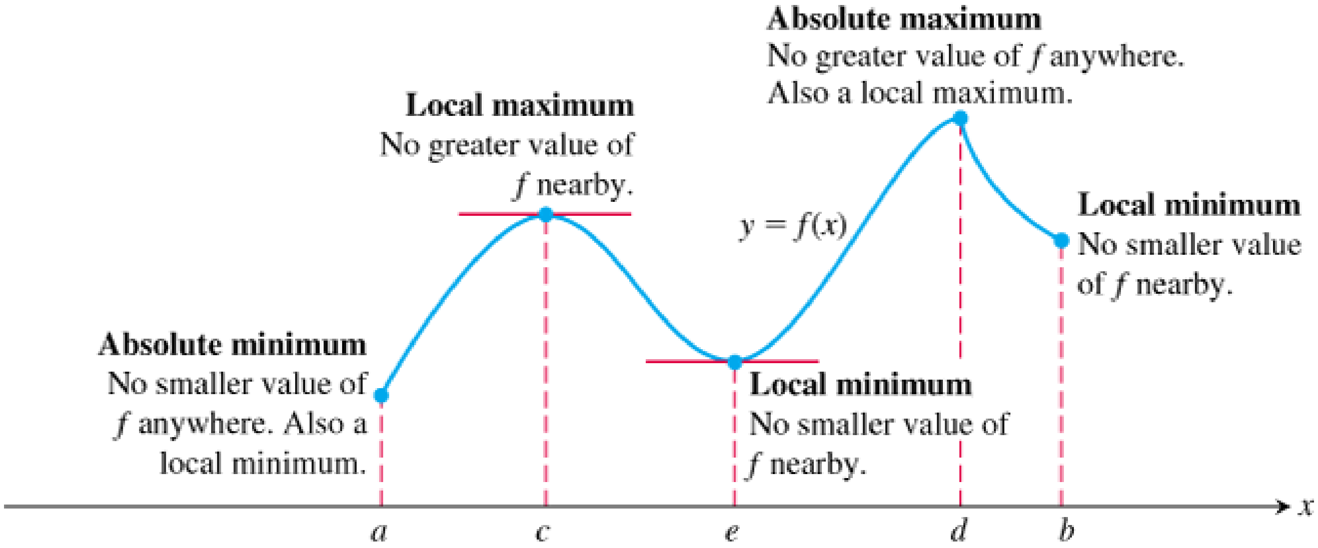
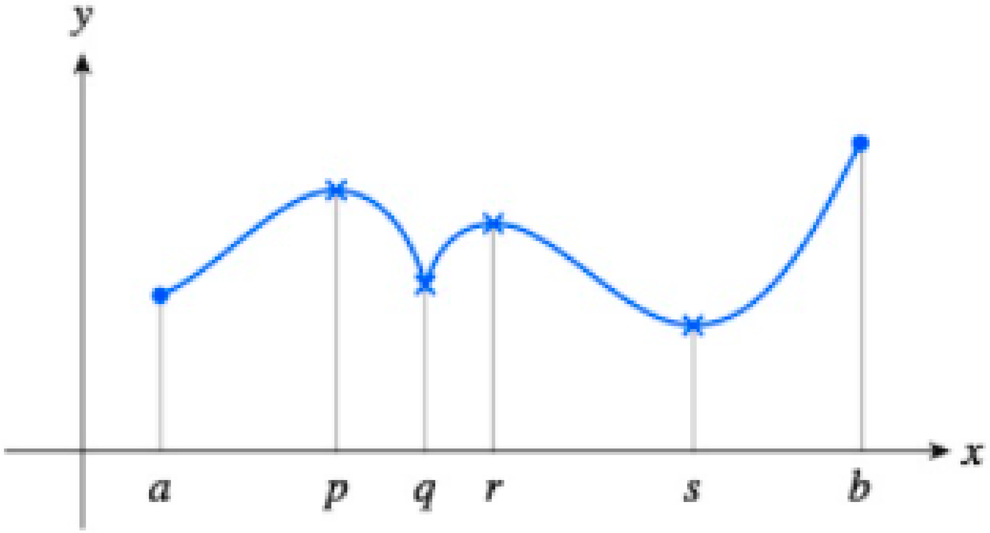


Figure 4.5

How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

Example 1

Identify and classify all extreme values of f.



KENNESAW STATE UNIVERSITY

Critical Numbers

- How do we find the local extrema?

- A number c is called a **critical number** of f if c is in the domain of f and

i. $f'(c) = 0$, or

ii. $f'(c)$ does not exist.

*Not all critical numbers will give us a local extremum (we have to check), but local extrema *cannot* occur anywhere else!

KENNESAW STATE UNIVERSITY

Concept Check

True or False:

Every critical number leads to a local extrema.

Example 2

Find the critical numbers of $f(x) = x^3 - 9x^2 + 24x - 10$.

Example 3

Find the critical numbers of $f(x) = |x|$.

Example 4

Find the critical numbers of $f(x) = x^2 - 32\sqrt{x}$.

Fermat's Theorem

Theorem: Fermat's Theorem on Local Extrema
If $f(c)$ is a local min or max, then c is a critical number of f.

Proof: Suppose that $f(c)$ is a local minimum (the case of a local maximum is similar). If $f'(c)$ does not exist, then c is a critical point and there is nothing more to prove. So, assume $f'(c)$ exists. We must then prove that $f'(c) = 0$.

Because $f(c)$ is a local minimum, we have $f(c+h) \geq f(c)$ for all sufficiently small $h \neq 0$. Equivalently, $f(c+h) - f(c) \geq 0$. Now divide this inequality by h. If $h > 0$, then $\frac{f(c+h)-f(c)}{h} \geq 0$; if $h < 0$, then $\frac{f(c+h)-f(c)}{h} \leq 0$. Thus, $f'(c) = 0$ by the Squeeze Theorem. □

KENNESAW STATE UNIVERSITY

Fermat’s Theorem (2)

- CAUTION: The converse does not hold. That is, if $x = c$ is a critical number of f, then $f(c)$ is not necessarily a local max or min.
- Consider $y = x^3$.

KENNESAW STATE UNIVERSITY

Extreme Values on a Closed Interval

- The combination of the Extreme Value Theorem and Fermat's Theorem tells us exactly where to find *absolute extrema* over a closed interval.

Theorem: Extreme Values on a Closed Interval

Assume that f is continuous on $[a, b]$ and let $f(c)$ be the minimum or maximum of f on $[a, b]$. Then c is a critical number or one of the endpoints a or b.

Closed Interval Method

Finding the Absolute Extrema for a Continuous Function f on a Finite Closed Interval $[a, b]$:

1. Find all critical values c of f on the interval $[a, b]$.
2. Evaluate f for all critical numbers c *and* the endpoints, a and b.
3. Make a conclusion: the *absolute minimum* is the smallest of these y-values, and the *absolute maximum* is the largest of these y-values.

KENNESAW STATE UNIVERSITY

Example 5

Find the absolute extrema $f(x) = 2x^3 - 15x^2 + 24x + 7$ over $[0, 6]$.

Example 6

Find the absolute extrema $g(x) = \sqrt{4 - x^2}$ over $[-1, 2]$.

KENNESAW STATE UNIVERSITY

Solo Practice 4.2

Find the absolute extrema of $f(x) = 1 - (x-1)^{2/3}$ over $[-1, 2]$.