September 10 Math 3260 sec. 51 Fall 2025

Chapter 2 Systems of Linear Equations

System of Linear Equations

A **system of linear equations** (a.k.a. a *linear system*) is a collection of one or more linear equations in the same variables considered together. A generic system of m equations in n variables is

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}. \qquad (3)$$

If $b_i = 0$ for every $i = 1, \dots, m$ the system is **homogeneous**, otherwise it is **nonhomogeneous**.

A **solution** is as an ordered n-tuple of real numbers, $(s_1, s_2, \dots, s_n)$, having the property that upon substitution, $x_1 = s_1, \quad x_2 = s_2, \quad \cdots, \quad x_n = s_n$, every equation in the system reduces to an identity. The collection of all solutions of is called the **solution set** of the system.

Definition: Equivalent Systems

We will say that two systems of linear equations are **equivalent** if they have the same solution set.

Existence & Uniqueness

Theorem: For a system of linear equations, exactly one of the following is true:

i. the solution set is empty (i.e., there is no solution),
ii. there exists a unique solution, or
iii. there are infinitely many solutions.

The system is called *inconsistent* if it has no solutions, otherwise it is called **consistent**

Homogeneous systems are always consistent, because they always admit the **trivial solution**, $\vec{x} = \vec{0}_n$.

2.1.1 Systems of Two Equations with Two Variables

A system of two equations in two variables has the form

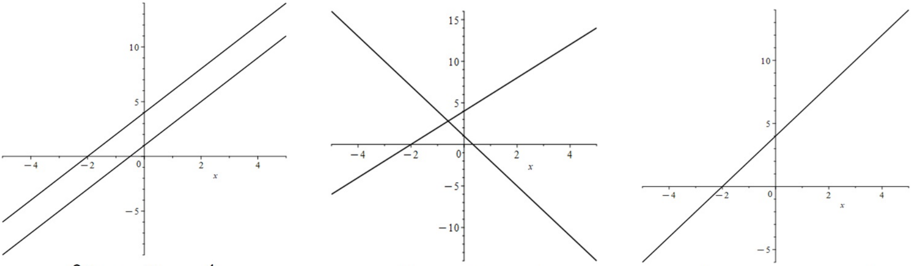
$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}.$$

We can think of the two equations as corresponding to a pair of lines, something like

$$x_2 = (\text{slope})\, x_1 + (\text{intercept}).$$

Such systems allow us to compare the three solution cases geometrically.

The three solution cases are easily visualized in R^2.



$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 2x_1 & - & x_2 & = & 1 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 3x_1 & + & x_2 & = & -1 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ -4x_1 & + & 2x_2 & = & -8 \end{array}$$

Figure: Lines determined by two linear equations in two variables illustrating the three possible geometric relationships.

i. parallel, non-intersecting lines correspond to an inconsistent system,
ii. lines with two distinct slopes correspond to a system with one solution,
iii. coincident lines corresponds to a system with infinitely many solutions.

3 Equations in 3 Variables

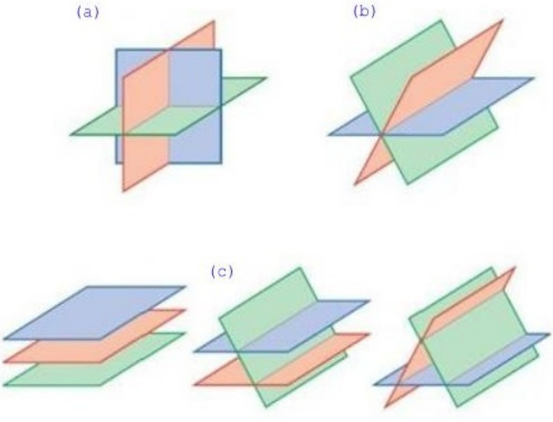


Figure: The graph of $a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

2.2 Solving a System of Linear Equations

Consider the pair of systems

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & - & 3x_2 & & & = & -7 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \\ & & & & x_3 & = & 3 \end{array}$$

These systems are **equivalent** (this is NOT obvious). We can get the solution to the system on the right by a little back substitution.

Note that the system on the right has a triangular structure. In particular, x_1 only appears in the first equation. x_2 appears in equations 1 and 2; x_3 appears in equations 1, 2, and 3. To the extent possible, this is what we're after.

Operations on Systems

There are three operations we can perform on a system that results in an equivalent system.

Elementary Equation Operations

- Multiply an equation by a nonzero scalar k. (scale)
- Interchange the position of any two equations. (swap)
- Replace an equation with the sum of itself and a multiple of any other equation. (replace)

When we add two equations, we add **like terms**.

Some Notation

We'll use E_i to refer to the i^{th} equation at any step in the solution process. Symbolically, we can write each operation:

- (scale) Replace equation i with k times itself: $kE_i \to E_i$
- (swap) Interchange equations i and j: $E_i \leftrightarrow E_j$
- (replace) Replace equation j with the sum of k times equation i plus equaiton j: $kE_i + E_j \to E_j$

Scale

To indicate that we are scaling equation E_i by the nonzero factor k, we'll write

$$kE_i \to E_i$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_3 \to E_3} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -12 \end{array}$$

Swap

To indicate that we are swapping equations E_i and E_j, we'll write

$$E_i \leftrightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{E_1 \leftrightarrow E_3} \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

Replace

To indicate that we are replacing equation E_j with the sum of itself and k times equation E_i, we'll write

$$kE_i + E_j \to E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_1 + E_2 \to E_2} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Note

$$\begin{array}{lrcrcrcr} & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ & 2x_1 & & & + & x_3 & = & 7 \\ \hline \text{(add)} & 0x_1 & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

Gaussian Elimination

We'll use some sequence of the three equation operations. Our goal is to change a system that looks something like

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

into one that looks something like

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array} .$$

If we can do that, we can use back substitution to find x_2, then x_1 and identify the solution.

Example

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

First goal: get x_1 in only the top equation using operations like $kE_1+E_2 \to E_2$ to get rid of x_1 in the second and third equations. We have to pick the right k's.

$E_1 \leftrightarrow E_3$

Step 1 →

$$\begin{array}{l} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = -4 \\ 2x_1 \quad\quad + x_3 = 7 \end{array}$$

This is easier if the coef. of x_1 is 1, so we swap E_1 and E_3.

$-E_1 + E_2 \to E_2$

Step 2 →

$$\begin{array}{l} x_1 + x_2 + x_3 = 6 \\ \quad\quad x_2 - 2x_3 = -10 \\ 2x_1 \quad\quad + x_3 = 7 \end{array}$$

$-2E_1 + E_3 \to E_3$

step 3 →

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ -2x_2 - x_3 &= -5 \end{aligned}$$

$$\begin{aligned} -2x_1 - 2x_2 - 2x_3 &= -12 \\ 2x_1 \quad + x_3 &= 7 \end{aligned}$$

We got rid of the x_1's, now we move on to get rid of x_2 below the second equation.

$2E_2 + E_3 \to E_3$

Step 4 →

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ -5x_3 &= -25 \end{aligned}$$

$$\begin{aligned} 2x_2 - 4x_3 &= -20 \\ -2x_2 - x_3 &= -5 \end{aligned}$$

$-\frac{1}{5}E_3 \to E_3$

Step 5 →

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned}$$

$x_3 = 5$, back substitution

$$x_2 = -10 + 2x_3 = -10 + 2(5) = 0$$

$$x_1 = 6 - x_2 - x_3 = 6 - 0 - 5 = 1$$

We found exactly one solution

$x_1 = 1$

$x_2 = 0$

$x_3 = 5$

parametric description

$\vec{x} = (1, 0, 5)$ vector

parametric.

If we find exactly one number for each variable, then the system has exactly one solution.

On the next slides, we'll see what the no-solution and infinitely many solutions cases look like.

Other Solution Cases

An example of the no solution case: The system

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

Inconsistent systems always give rise to an equation that is false.

$$0 = \text{something nonzero}$$

Other Solution Cases

An example of an infinite solutions case: The system

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \\ -2x_1 & + & 2x_2 & + & 10x_3 & = & -12 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & -2 \\ & & & & 0 & = & 0 \end{array}$$

Consistent systems with infinitely many solutions are always the result when

1. there are no false statements, and
2. there are more variables than nontrivial equations.