September 10 Math 3260 sec. 53 Fall 2025

Chapter 2 Systems of Linear Equations

System of Linear Equations

A **system of linear equations** (a.k.a. a *linear system*) is a collection of one or more linear equations in the same variables considered together. A generic system of m equations in n variables is

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}. \qquad (3)$$

If $b_i = 0$ for every $i = 1, \dots, m$ the system is **homogeneous**, otherwise it is **nonhomogeneous**.

A **solution** is as an ordered n-tuple of real numbers, $(s_1, s_2, \dots, s_n)$, having the property that upon substitution, $x_1 = s_1, \quad x_2 = s_2, \quad \cdots, \quad x_n = s_n$, every equation in the system reduces to an identity. The collection of all solutions of is called the **solution set** of the system.

Definition: Equivalent Systems

We will say that two systems of linear equations are **equivalent** if they have the same solution set.

Existence & Uniqueness

Theorem: For a system of linear equations, exactly one of the following is true:

i. the solution set is empty (i.e., there is no solution),
ii. there exists a unique solution, or
iii. there are infinitely many solutions.

The system is called *inconsistent* if it has no solutions, otherwise it is called **consistent**

Homogeneous systems are always consistent, because they always admit the **trivial solution**, $\vec{x} = \vec{0}_n$.

2.1.1 Systems of Two Equations with Two Variables

A system of two equations in two variables has the form

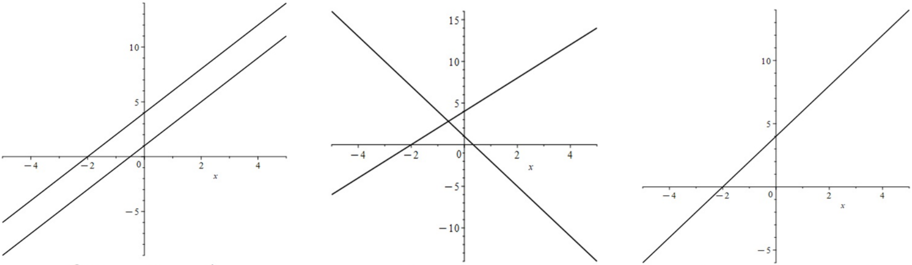
$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}.$$

We can think of the two equations as corresponding to a pair of lines, something like

$$x_2 = (\text{slope})\, x_1 + (\text{intercept}).$$

Such systems allow us to compare the three solution cases geometrically.

The three solution cases are easily visualized in R^2.



$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 2x_1 & - & x_2 & = & 1 \end{array}$$

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 3x_1 & + & x_2 & = & -1 \end{array}$$

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ -4x_1 & + & 2x_2 & = & -8 \end{array}$$

Figure: Lines determined by two linear equations in two variables illustrating the three possible geometric relationships.

i. parallel, non-intersecting lines correspond to an inconsistent system,
ii. lines with two distinct slopes correspond to a system with one solution,
iii. coincident lines corresponds to a system with infinitely many solutions.

3 Equations in 3 Variables

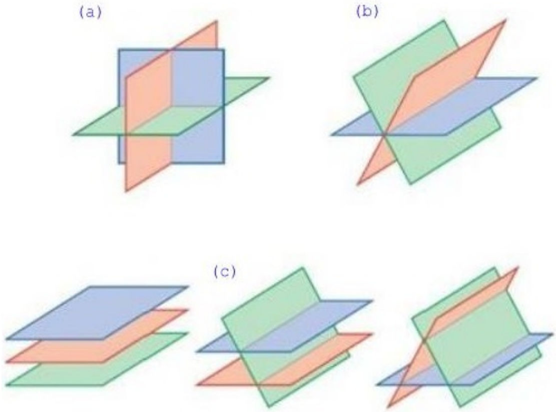


Figure: The graph of $a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

2.2 Solving a System of Linear Equations

Consider the pair of systems

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & - & 3x_2 & & & = & -7 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \\ & & & & x_3 & = & 3 \end{array}$$

These systems are **equivalent** (this is NOT obvious). We can get the solution to the system on the right by a little back substitution.

Note that the system on the right has a triangular structure. In particular, x_1 only appears in the first equation. x_2 appears in equations 1 and 2; x_3 appears in equations 1, 2, and 3. To the extent possible, this is what we're after.

Operations on Systems

There are three operations we can perform on a system that results in an equivalent system.

Elementary Equation Operations

- Multiply an equation by a nonzero scalar k. (scale)
- Interchange the position of any two equations. (swap)
- Replace an equation with the sum of itself and a multiple of any other equation. (replace)

When we add two equations, we add **like terms**.

Some Notation

We'll use E_i to refer to the i^{th} equation at any step in the solution process. Symbolically, we can write each operation:

- (scale) Replace equation i with k times itself: $kE_i \to E_i$
- (swap) Interchange equations i and j: $E_i \leftrightarrow E_j$
- (replace) Replace equation j with the sum of k times equation i plus equaiton j: $kE_i + E_j \to E_j$

Scale

To indicate that we are scaling equation E_i by the nonzero factor k, we'll write

$$kE_i \to E_i$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_3 \to E_3} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -12 \end{array}$$

Swap

To indicate that we are swapping equations E_i and E_j, we'll write

$$E_i \leftrightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{E_1 \leftrightarrow E_3} \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

Replace

To indicate that we are replacing equation E_j with the sum of itself and k times equation E_i, we'll write

$$kE_i + E_j \to E_j$$

For example

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_1 + E_2 \to E_2} \quad \begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Note

$$\begin{array}{cccccccc} & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ & 2x_1 & & & + & x_3 & = & 7 \\ \hline \text{(add)} & 0x_1 & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

Gaussian Elimination

We'll use some sequence of the three equation operations. Our goal is to change a system that looks something like

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

into one that looks something like

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}.$$

If we can do that, we can use back substitution to find x_2, then x_1 and identify the solution.

Example

$$\begin{aligned} 2x_1 \quad\quad\quad + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -4 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

First goal: get x_1 in only the top equation using operations like $kE_1+E_2 \to E_2$ to get rid of x_1 in the second and third equations. We have to pick the right k's.

This is easier if the coef. of x_1 is 1, so we swap E_1 and E_3.

$E_1 \leftrightarrow E_3$

step 1

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 - x_3 &= -4 \\ 2x_1 \quad\quad + x_3 &= 7 \end{aligned}$$

$-E_1 + E_2 \to E_2$

step 2

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ 2x_1 \quad\quad + x_3 &= 7 \end{aligned}$$

$-2E_1 + E_3 \to E_3$

step 3

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ -2x_2 - x_3 &= -5 \end{aligned}$$

$-2x_1 - 2x_2 - 2x_3 = -12$

$2x_1 \quad + x_3 = 7$

$2E_2 + E_3 \to E_3$

step 4

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ -5x_3 &= -25 \end{aligned}$$

$2x_2 - 4x_3 = -20$

$-2x_2 - x_3 = -5$

$-\frac{1}{5} E_3 \to E_3$

step 5

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned}$$

Back substitution

$x_3 = 5$

$x_2 = -10 + 2x_3 = -10 + 2(5) = 0$

$x_1 = 6 - x_2 - x_3 = 6 - 0 - 5 = 1$

Parametric description.

$x_1 = 1$

$x_2 = 0$

$x_3 = 5$

In vector parametric

$\vec{x} = \langle 1, 0, 5 \rangle.$

What we saw happen here, where we found exactly one number for each variable, is representative of systems having exactly one solution.

On the next slides, we'll see examples of what happens when there are no solutions or when there are infinitely many solutions.

Other Solution Cases

An example of the no solution case: The system

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

Inconsistent systems always give rise to an equation that is false.

$$0 = \text{something nonzero}$$

Other Solution Cases

An example of an infinite solutions case: The system

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \\ -2x_1 & + & 2x_2 & + & 10x_3 & = & -12 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & -2 \\ & & & & 0 & = & 0 \end{array}$$

Consistent systems with infinitely many solutions are always the result when

1. there are no false statements, and
2. there are more variables than nontrivial equations.