September 11 Math 2306 sec. 51 Fall 2024

Section 5: First Order Equations: Models and Applications

We are considering another population model, one that accounts for environmental limitations on a population.

Logistic Growth Model

The equation $\frac{dP}{dt} = kP(M-P)$, where $k, M > 0$ is called a **logistic growth equation**.

Suppose the intial population $P(0) = P_0$. Solve the resulting initial value problem. Show that if $P_0 > 0$, the population tends to the carrying capacity M.

Logistic Growth: $P'(t) = kP(M - P)\ P(0) = P_0$

We recognized the equation as being both separable and Bernoulli. We solved it as a Bernoulli equation and obtained the one-parameter family of solutions

$$P(t) = \frac{M}{1 + Ae^{-kMt}}.$$

We need to apply the initial condition.

$$P(0) = \frac{M}{1 + A e^0} = \frac{M}{1 + A} = P_0$$

$$\Rightarrow M = (1 + A) P_0 = P_0 + P_0 A$$

$$P_0 A = M - P_0 \Rightarrow A = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + \left(\frac{M - P_0}{P_0}\right) e^{-kMt}}$$

$$= \frac{M}{1 + \left(\frac{M - P_0}{P_0}\right) e^{-kMt}} \left(\frac{P_0}{P_0}\right)$$

$$P(t) = \frac{M P_0}{P_0 + (M - P_0) e^{-kMt}}$$

If $P_0 > 0$,

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{M P_0}{P_0 + (M - P_0) e^{-kMt}}$$

$$= \frac{M P_0}{P_0 + 0} = \frac{M P_0}{P_0} = M$$

$P \to M$ as $t \to \infty$.

Logistic Modeling

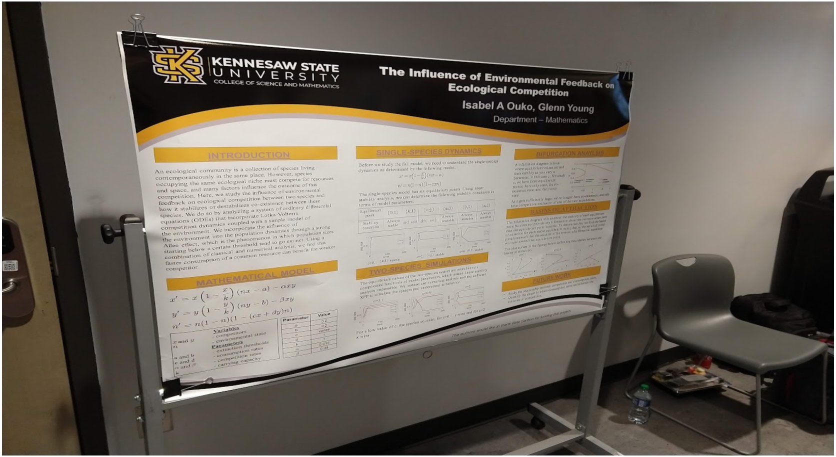
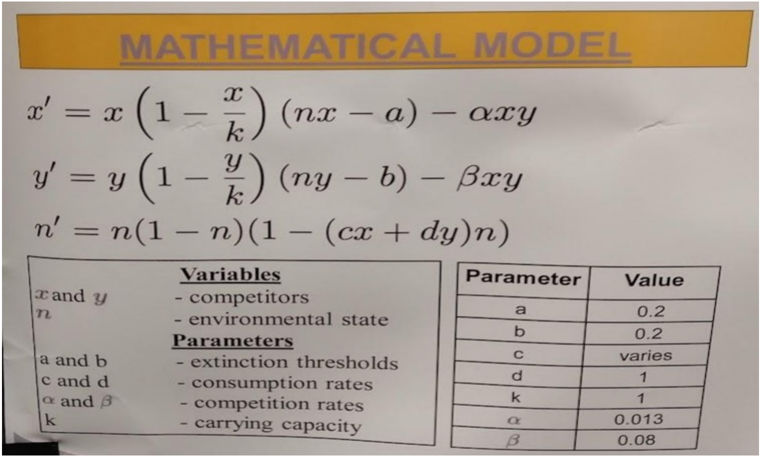


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling



MATHEMATICAL MODEL

$$x' = x\left(1 - \frac{x}{k}\right)(nx - a) - \alpha xy$$

$$y' = y\left(1 - \frac{y}{k}\right)(ny - b) - \beta xy$$

$$n' = n(1 - n)(1 - (cx + dy)n)$$

Variables

x and y	- competitors
n	- environmental state

Parameters

a and b	- extinction thresholds
c and d	- consumption rates
α and β	- competition rates
k	- carrying capacity

Parameter	Value
a	0.2
b	0.2
c	varies
d	1
k	1
α	0.013
β	0.08

Figure: The species equations include an extended logistic term with threshold and competition.

Long Time Solution of Logistic Equation

$$\frac{dP}{dt} = kP(M-P) = -kP^2 + kMP.$$

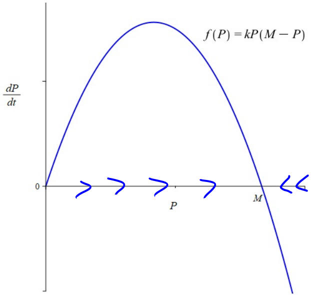


Figure: Plot of P versus $\frac{dP}{dt}$. Note that $\frac{dP}{dt} > 0$ if $0 < P < M$ and $\frac{dP}{dt} < 0$ if $P > M$.

Logistic Growth with Breeding Threshhold

Suppose we modify the logistic equation based on the assumption that the fish will only breed successfully if the population is above some minimum threshhold N where $0 < N < M$. The new model is

$$\frac{dP}{dt} = kP(M-P)(P-N),$$

which is separable. But it is practically impossible to obtain an explicit solution, $P(t) =$ "some function of t."

Remark: Even without solving the equation, we can use the equation to predict the long time solution based on the initial population.

Solutions are given implicitly by $\dfrac{(M-P)^N}{P^{M-N}(P-N)^M} = Ae^{-kMN(M-N)t}$.

Expected Long Time Solutions

$$\frac{dP}{dt} = kP(M - P)(P - N),$$

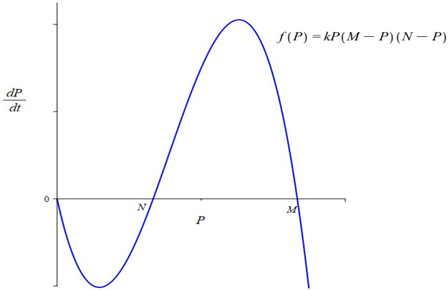
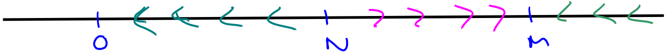


Figure: Plot of P versus $\frac{dP}{dt}$ for the modified model.

Expected Long Time Solutions

Use the given plot of $F(P) = kP(M - P)(P - N)$ to determine the long time solution of $\dfrac{dP}{dt} = F(P)$ if (a) $0 < P(0) < N$, (b) $N < P(0) < M$ or (c) $P(0) > M$.



(a) $0 < P(0) < N$ $\quad \lim_{t \to \infty} P(t) = 0$ extinction

(b) $N < P(0) < M$ $\quad \lim_{t \to \infty} P(t) = M$

(c) $P(0) > M$, $\quad \lim_{t \to \infty} P(t) = M$

Qualitative Analysis

Autonomous Equation

The differential equation $\frac{dy}{dt} = f(t, y(t))$ is called **autonomous** if the right hand side does not depend explicitly on t—i.e., an autonomous equation has the form

$$\frac{dy}{dt} = F(y).$$

$$\frac{dy}{dt} = y(2-y)(y+3)$$ autonomous

$$\frac{dy}{dt} = t^2(y-3)$$ non autonomous

Equilibrium Solutions

Equilibrium Solutions

If y_0 is a value such that $F(y_0) = 0$, then the constant function $y(t) = y_0$ is called an **equilibrium** solution (or equilibrium point) of the autonomous differential equation $\frac{dy}{dt} = F(y)$.

Note: If $y(0) = y_0$ and y_0 is an equilibrium solution, then $y(t) = y_0$ is a constant solution.

Question: What if $y(0)$ is not an equilibrium value, but is close to an equilibrium value? What can we expect from the solution?

Stability of Equilibrium Solutions

In general, we may classify an equilibrium solution of a given autononous ODE as being

- **unstable**: solutions close, but not exact, will tend away from the equilibrium value,
- **stable**: solutions close, but not exact, will tend towards the equilibrium value[1], or
- **semi-stable**: solutions close, but not exact, may tend towards or away from the equilibrium value depending on whether the solution is greater than or less than the equilibrium value.

Note: There are more detailed notions of stability, so there's more to the story. But we'll consider the above definitions here.

[1] This is more accurately referred to as asymptotically stable

Determining Stability of an Equilibrium Solution

To determine the nature of an equilibrium solution y_0 for an ODE $y' = F(y)$, we can analyze the sign of F in the neighborhood of y_0. Suppose F is continuous on an open interval about y_0.

- If F changes sign from positive (+) to negative (-) as y passes through y_0 (from left to right), then y_0 is a stable equilibrium.
- If F changes signs from negative (-) to positive (+) as y passes through y_0, then then y_0 is an unstable equilibrium.
- If F doesn't change signs, then y_0 is semi-stable.

If this reminds you of a *derivative test*, there's a good reason for that. Fortunately, it's easy to visualize the cases if you can obtain even a crude drawing of the graph of F.

Example

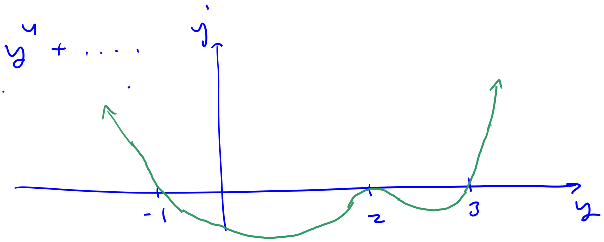
Consider the IVP

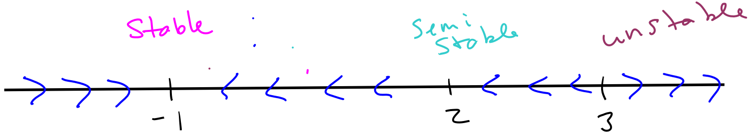
$$y' = 2(y+1)(2-y)^2(y-3), \quad y(0) = k.$$

Determine the long time behavior, $\lim_{t\to\infty} y(t)$, if

(a) $k = -2$,	(b) $k = 0$,	(c) $k = 1$,
(d) $k = 2$,	(e) $k = 2.5$,	(f) $k = 4$.

$F(y) = 2y^4 + \ldots$





(a) $k = -2$, (b) $k = 0$, (c) $k = 1$,
(d) $k = 2$, (e) $k = 2.5$, (f) $k = 4$.

a) $y(0) = -2$ $\lim_{t \to \infty} y(t) = -1$

b) $y(0) = 0$ $\lim_{t \to \infty} y(t) = -1$

c) $y(0) = 1$ $\lim_{t \to \infty} y(t) = -1$

d) $y(0) = 2$ $\lim_{t \to \infty} y(t) = 2$ $y(t) = 2$

e) $y(0) = 2.5$, $\lim_{t \to \infty} y(t) = 2$

f) $y(0) = 4$ $\lim_{t \to \infty} y(t) = \infty$

Models Derived in this Section

We have several models involving first order ODEs.

Exponential Growth/Decay

$$\frac{dP}{dt} = kP$$

RC-Series Circuit

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

LR-Series Circuit

$$L\frac{di}{dt} + Ri = E(t)$$

Classical Mixing

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}$$

Logistic Growth

$$\frac{dP}{dt} = kP(M - P)$$