September 11 Math 2306 sec. 52 Spring 2023

Section 5: First Order Equations: Models and Applications

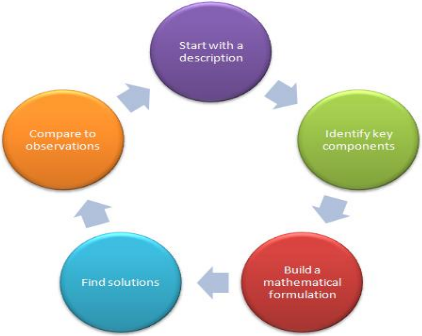


Figure: Mathematical Models give Rise to Differential Equations

In this section, we will consider select models involving first order ODEs. Let's see the process in action.

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2021, there were 58 rabbits. In 2022, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2031.

We need variables. The population is changing in time, so let's introduce variables

$$t \sim \text{time} \quad \text{and} \quad P(t) \sim \text{is the population (density) at time } t.$$

We need to express the following mathematically:

The population's rate of change is proportional to the population.

rate of change = $\frac{dP}{dt}$

$\frac{dP}{dt} \propto P$ ($\propto$: proportional to) $\implies$ $\frac{dP}{dt} = kP$ for some constant k.

$\frac{dP}{dt} = kP$ is an ODE for the population P.

Let's take t in years w/ $t=0$ in 2021.

The data becomes $P(0) = 58$ and $P(1) = 89$

Let's solve the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

The ODE is separable and linear. Separating.

$$\frac{1}{P}\frac{dP}{dt} = k \implies \frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt \implies \ln|P| = kt + C$$

Exponentiate $|P| = e^{kt+c} = e^{c}e^{kt}$, let $A = \pm e^{c}$ or zero

$$P = Ae^{kt}.$$ Apply $P(0) = 58$

$$P(0) = Ae^{0} = 58 \implies A = 58,$$ the population

$$P(t) = 58e^{kt}.$$ We can find k from knowing that $P(1) = 89$. Hence

$$P(1) = 58e^{k(1)} = 89 \implies e^{k} = \frac{89}{58}$$

$$\implies k = \ln\left(\frac{89}{58}\right).$$

This model says the population at time t is

$$P(t) = 58e^{t\ln\left(\frac{89}{58}\right)}$$

The year 2031 corresponds to $t=10$.

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198$$

The model predicts a population of roughly 4200 rabbits in 2031.

Exponential Growth or Decay

Exponential Growth/Decay

If a quantity P changes continuously at a rate proportional to its current value, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear.

If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

In practice, we typically take $k > 0$ and in the case of decay, we write $\frac{dP}{dt} = -kP$.

Series Circuits: RC-circuit

With the restriction that we are considering only models involving first order equations, we can consider two types of simple circuits, an *RC*-series circuit or an *LR*-series circuit.

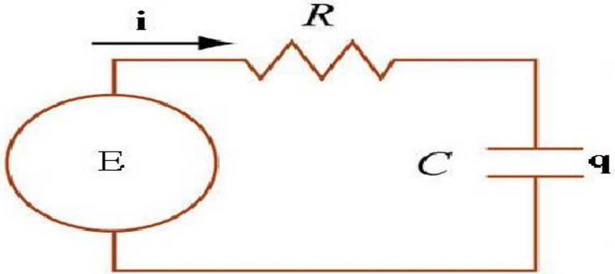


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge on the capacitor is q and the current $i = \frac{dq}{dt}$. Both q and i are functions of time.

Series Circuits: LR-circuit

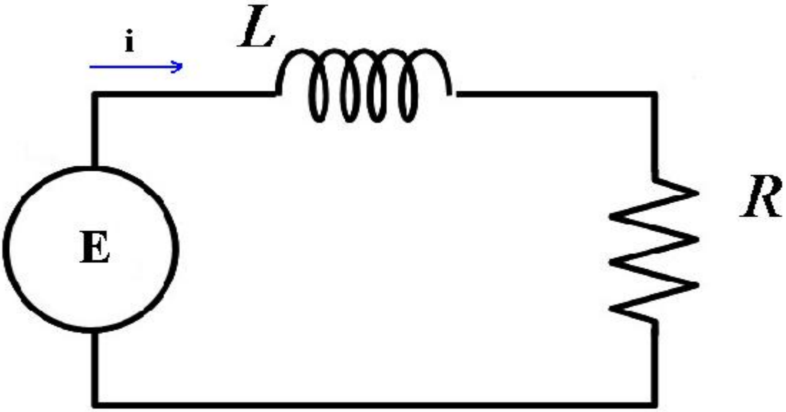


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. We track the current i as a function of time.

Measurable Quantities:

In these problems, there are several measurable quantities. These are listed here along with the relevant units of measure.

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Table: The potential drop across various elements is know empirically.

Kirchhoff’s Law

Kirchhoff’s Law

Kirchhoff’s Law states that:
The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force. We can use this to arrive at a differential equation for the charge $q(t)$ in an RC circuit or the current $i(t)$ in and LR circuit.

Both of these result in a first order linear differential equation.

RC Series Circuit

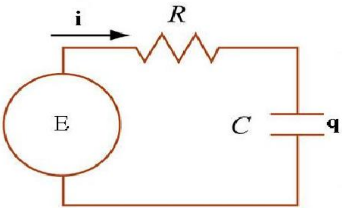


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

drop across resistor	$+$	drop across capacitor	$=$	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

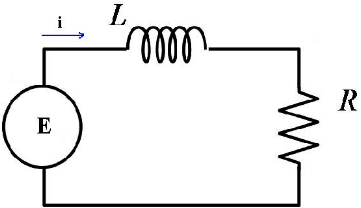


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Summary of First Order Circuit Models

Before considering an example, let's summarize our two circuit models.

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0)=0.4A$. Determine the charge as $t\to\infty$.

The RC equation is $R\frac{dq}{dt}+\frac{1}{C}q=E$.

Here, $E(t)=200$ V, $R=1000\Omega$ and $C=5\cdot 10^{-6}$ f

$1000\frac{dq}{dt}+\frac{1}{5\cdot 10^{-6}}q=200$, $i(0)=q'(0)=0.4$

$\frac{1}{5\cdot 10^{-6}}=\frac{10^6}{5}=\frac{10\cdot 10^5}{5}=2\cdot 10^5$, and $2\cdot 10^5\div 1000=\frac{2\cdot 10^5}{10^3}=200$

In standard form, the ODE is

$$\frac{dq}{dt} + 200q = \frac{1}{5} \quad , \quad q'(0) = \frac{2}{5}$$

$$P(t) = 200 \quad , \quad \mu = e^{\int p(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} q\right) dt = \int \frac{1}{5} e^{200t}\, dt$$

$$e^{200t} q = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + k$$

$$\Rightarrow \quad q(t) = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = \frac{2}{5}$

$$q'(t) = -200\,k\,e^{-200t}, \quad q'(0) = -200\,k\,e^{0} = \frac{2}{5}$$

$$k = \frac{2}{5(-200)} = \frac{-1}{500}$$

The charge on the capacitor for all $t > 0$ is

$$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}$$

The steady state charge $q_p = \frac{1}{1000}$ and the transient state charge $q_c = \frac{-1}{500}e^{-200t}$.

The long term charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \frac{1}{1000} - \frac{1}{500}e^{-200t} = \frac{1}{1000}$$

Coulombs

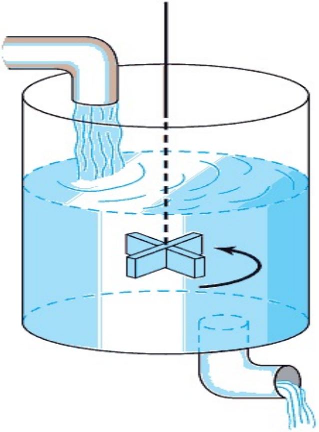
A Classic Mixing Problem

Classical mixing involves tracking the mass of some substance in a composite mixture. Examples include

- salt in a salt-water mixture,
- ethanol in an ethanol-gasoline mixture,
- polutant in a volume of water.

Let's look at a specific problem and build a model that can be used in general. First, a visual.

A Classic Mixing Problem



A composite fluid is kept *well mixed* (i.e. spatially homogeneous).

Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

We'll derive a model and solve this problem next time.

Some Notation

In addition to the amount of salt, $A(t)$, at time t we have several variables or parameters. Let

- r_i be the rate at which fluid enters the tank (rate in),
- r_o be the rate at which fluid leaves the tank (rate out),
- c_i be the concentration of substance (salt) in the in-flowing fluid (concentration in),
- c_o be the concentration of substance (salt) in the out-flowing fluid (concentration out),
- $V(t)$ be the total volume of fluid in the tank at time t,
- V_0 be the volume of fluid in the tank at time $t = 0$, i.e., $V_0 = V(0)$

A Classic Mixing Problem Illustrated

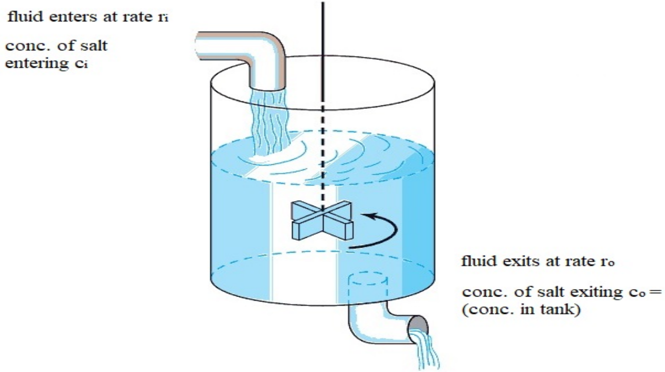


Figure: Values for c_i, r_i, and r_o are given in the problem statement. The well mixed assumption means that c_o will match the concentration in the tank.

This means that c_o is **NOT constant**! It depends on time through both A and V.