September 12 Math 2306 sec. 51 Fall 2022

Section 5: First Order Equations Models and Applications

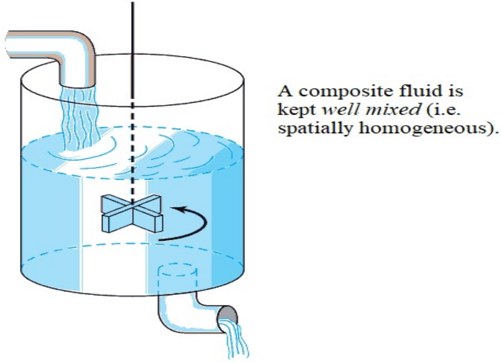

Figure: We were considering a **classic mixing problem** in which we track the quantity of an element in a spatially uniform composite fluid with inflow and outflow.

Classic Mixing Problem Model

The mass, $A(t)$, of a substance in a composite fluid that is well mixed satisfies the equation

$$\frac{dA}{dt} = r_i c_i - r_o c_o, \quad \text{subject to} \quad A(0) = A_0$$

where

- r_i and r_o are the rates at which fluid flows in, respectively out, of the receptacle,
- c_i is the concentration of the substance in the in-flowing fluid,
- $c_o = \dfrac{A(t)}{V(t)}$ is the concentration of the substance in the out-flowing fluid,
- $V(t) = V(0) + (r_i - r_o)t$ is the volume of fluid in the tank at time t, and
- A_0 is the initial mass of the substance.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We have $r_i = r_o = 5$, $c_i = 2$, $A(0) = 0$ and

$$c_o = \frac{A}{V} = \frac{A}{500}.$$

So A satisfies the IVP

$$\frac{dA}{dt} = 5(2) - 5\frac{A}{500}, \quad A(0) = 0.$$

This equation is both first order linear and separable. In standard form, the linear equation is

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

$$\frac{dy}{dt} + P(t)\,y = f(t) \qquad \text{Here, } P(t) = \frac{1}{100}$$

$$\mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

Multiply by μ

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\,e^{\frac{1}{100}t}$$

Integrate $\displaystyle\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\,e^{\frac{1}{100}t}\,dt$

$$e^{\frac{1}{100}t} A = 10\left(\frac{1}{1/100}\right) e^{\frac{1}{100}t} + C$$

$$e^{\frac{1}{100}t} A = 1000\,e^{\frac{1}{100}t} + C$$

$$A = \frac{1000\, e^{\frac{1}{100}t} + C}{e^{1/100\, t}} = 1000 + C e^{-\frac{1}{100}t}$$

A 1parameter family of solutions to the ODE.

Apply $A(0)=0$.

$$A(0) = 1000 + C e^{0} = 0$$

$$\Longrightarrow \quad C = -1000$$

$$\boxed{\begin{array}{l}\text{The amount of salt at the time } t \text{ is} \\ A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}\end{array}}$$

Let C_s be the concentration of salt in the tank @ $t=5$ min.

$$C_s = \frac{A(5)}{V(5)} = \frac{1000 - 1000e^{\frac{-1}{100}(5)}}{500 \text{ gal}} \text{ lb} \approx 0.01 \ \frac{\text{lb}}{\text{gal}}$$

Side note: The concentration in the tank, C

$$C = \frac{A}{V} = \frac{1000 - 1000e^{\frac{-1}{100}t}}{500} \ \frac{\text{lb}}{\text{gal}}$$

As $t \to \infty$ $\quad C \to \frac{1000}{500} \ \frac{\text{lb}}{\text{gal}} = 2 \ \frac{\text{lb}}{\text{gal}}$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5 \frac{\text{gal}}{\text{min}}$, $c_i = 2 \frac{\text{lb}}{\text{gal}}$, $V(0) = 500$ gal

$r_o = 10 \frac{\text{gal}}{\text{min}}$

$$C_o = \frac{A}{V} = \frac{A}{500 + (5-10)t} = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i C_i - r_o C_o$$

$$= 5(2) - 10\left(\frac{A}{500 - 5t}\right)$$

$$\frac{dA}{dt} = 10 - 2\frac{A}{100-t}$$

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$ standard form

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

The quantities of interest are

- P current population
- $M - P$ difference between current population and M
- $\frac{dP}{dt}$ rate of change of population

$$\frac{dP}{dt} \propto P(M-P) \implies \frac{dP}{dt} = kP(M-P)$$

for constant k.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The equation is separable and Bernoulli.

Treating it as Bernoulli

$$\frac{dP}{dt} = kMP - kP^2$$

$$\Rightarrow \quad \frac{dP}{dt} - kMP = -kP^2$$

Bernoulli $\quad \frac{dy}{dt} + Q(t)y = f(t)y^n$

$n=2, \quad Q(t) = -kM, \quad f(t) = -k$

set $u = P^{1-n} = P^{1-2} = P^{-1}, \quad 1-n = -1$

$$\frac{du}{dt} + (1-n)Q(t)u = (1-n)f(t)$$

$$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$$Q_1(t) = kM, \quad \mu = e^{\int Q_1(t)\,dt} = e^{\int kM\,dt} = e^{kMt}$$

$$\frac{d}{dt}\left(e^{kMt} u\right) = k e^{kMt}$$

$$e^{kMt} u = \int k e^{kMt} \, dt$$

$$= \frac{k}{kM} e^{kMt} + C$$

$$e^{kMt} u = \frac{1}{M} e^{kMt} + C$$

$$u = \frac{\frac{1}{M} e^{kMt} + C}{e^{kMt}} = \frac{1}{M} + C e^{-kMt}$$

$$u = P^{-1} = \frac{1}{P} \Rightarrow P = u^{-1} = \frac{1}{u}$$

Hence

$$P = \frac{1}{\frac{1}{M} + Ce^{-kMt}} \cdot \frac{M}{M}$$

$$P(t) = \frac{M}{1 + CMe^{-kMt}}$$

Let $P(0) = P_0$. Apply this condition.

$$P(0) = P_0 = \frac{M}{1 + CMe^{0}} = \frac{M}{1 + CM}$$

solve for C

$$(1 + CM)P_0 = M$$

$$P_0 + CMP_0 = M \Rightarrow CMP_0 = M - P_0$$

$$CM = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + CM e^{-kMt}}.$$

The solution

$$P(t) = \frac{M}{1 + \left(\frac{M-P_0}{P_0}\right) e^{-kMt}} \cdot \frac{P_0}{P_0}.$$

$$\boxed{P(t) = \frac{M P_0}{P_0 + (M - P_0) e^{-kMt}}}$$

$$\lim_{t\to\infty} P(t) = \frac{M P_0}{P_0} = M$$