September 12 Math 2306 sec. 52 Fall 2022

Section 5: First Order Equations Models and Applications

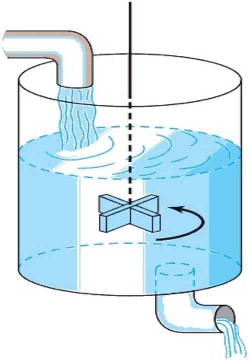


Figure: We were considering a **classic mixing problem** in which we track the quantity of an element in a spatially uniform composite fluid with inflow and outflow.

Classic Mixing Problem Model

The mass, $A(t)$, of a substance in a composite fluid that is well mixed satisfies the equation

$$\frac{dA}{dt} = r_i c_i - r_o c_o, \quad \text{subject to} \quad A(0) = A_0$$

where

- r_i and r_o are the rates at which fluid flows in, respectively out, of the receptacle,
- c_i is the concentration of the substance in the in-flowing fluid,
- $c_o = \dfrac{A(t)}{V(t)}$ is the concentration of the substance in the out-flowing fluid,
- $V(t) = V(0) + (r_i - r_o)t$ is the volume of fluid in the tank at time t, and
- A_0 is the initial mass of the substance.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We have $r_i = r_o = 5$, $c_i = 2$, $A(0) = 0$ and

$$c_o = \frac{A}{V} = \frac{A}{500}.$$

So A satisfies the IVP

$$\frac{dA}{dt} = 5(2) - 5\frac{A}{500}, \quad A(0) = 0.$$

This equation is both first order linear and separable. In standard form, the linear equation is

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

Solving the Mixing Problem

We solved this IVP using an integrating factor and obtained

$$A(t) = 1000 - 1000e^{-t/100}$$

Let's show that this equation is also separable.

$$\frac{dA}{dt} + \frac{1}{100}A = 10.$$

$\frac{dA}{dt} = g(t)h(A)$

$$\frac{dA}{dt} = 10 - \frac{1}{100}A$$

$$= \frac{1}{100}(1000 - A)$$

$g(t) = \frac{1}{100}$ and $h(A) = 1000 - A$

$$\frac{1}{1000-A}\frac{dA}{dt} = \frac{1}{100}$$

$$\int \frac{1}{1000-A}\, dA = \int \frac{1}{100}\, dt$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5\ \frac{\text{gal}}{\text{min}} \quad c_i = 2\ \frac{\text{lb}}{\text{gal}}, \quad r_o = 10\ \frac{\text{gal}}{\text{min}}$

$$c_o = \frac{A}{V} = \frac{A}{V(0) + (r_i - r_o)t} = \frac{A}{500 + (5-10)t}$$

$$= \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$$= 5(2) - 10\left(\frac{A}{500-5t}\right)$$

$$= 10 - \frac{2}{100-t}A$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

The quantities of interest are

- P current population
- $M - P$ difference between current population and M
- $\frac{dP}{dt}$ rate of change of population

$$\frac{dP}{dt} \propto P(M-P) \implies \frac{dP}{dt} = kP(M-P)$$

for constant k.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The ODE is separable and Bernoulli.

$$\frac{dP}{dt} = kMP - kP^2 \implies$$

$$\frac{dP}{dt} - kMP = -kP^2$$

$$\frac{dy}{dt} + Q(t)y = f(t)y^n$$ generic Bernoulli ODE

Here, $n=2$, $Q(t)=-kM$, $f(t)=-k$

Set $u = P^{1-n} = P^{1-2} = P^{-1}$, $1-n=-1$

$$\frac{du}{dt} + (1-n)\,Q(t)\,u = (1-n)\,f(t)$$

$$\frac{du}{dt} + (-1)(-kM)\,u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$Q_1(t) = kM$, $\mu = e^{\int Q_1(t)\,dt} = e^{\int kM\,dt} = e^{kMt}$

$$\frac{d}{dt}\left(e^{kMt}u\right) = k\,e^{kMt}$$

$$\int \frac{d}{dt}\left(e^{kMt}u\right)dt = \int k e^{kMt}\,dt$$

$$e^{kMt}u = k\left(\frac{1}{kM}\right)e^{kMt} + C$$

$$= \frac{1}{M}e^{kMt} + C.$$

$$\Rightarrow \quad u = \frac{\frac{1}{M}e^{kMt} + C}{e^{kMt}} = \frac{1}{M} + Ce^{-kMt}$$

$$u = P^{-1} \Rightarrow P = u^{-1} = \frac{1}{u}$$

The general solution

$$P(t) = \frac{1}{\frac{1}{M} + Ce^{-kMt}} \cdot \frac{M}{M}$$

$$P(t) = \frac{M}{1 + CMe^{-kMt}}$$

Suppose $P(0) = P_0 \neq 0$. Apply the IC

$$P(0) = P_0 = \frac{M}{1 + CMe^0} = \frac{M}{1 + CM}$$

Solving for CM

$$P_0(1 + CM) = M$$

$$P_0 + P_0CM = M$$

$$P_0CM = M - P_0 \Longrightarrow$$

$$CM = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + CM e^{-kMt}}, \quad CM = \frac{M - P_0}{P_0}$$

The population

$$P(t) = \frac{M}{1 + \left(\frac{M - P_0}{P_0}\right) e^{-kMt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M - P_0) e^{-kMt}}}$$

Note: if $P_0 = 0$, $P(t) = 0$ for all t.

If $P_0 > 0$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} = \frac{MP_0}{P_0}$$

$$= M \quad \text{as expected}$$

$$\frac{dP}{dt} = kP(M-P) \qquad k, M > 0$$

$0 < P_0 < M$

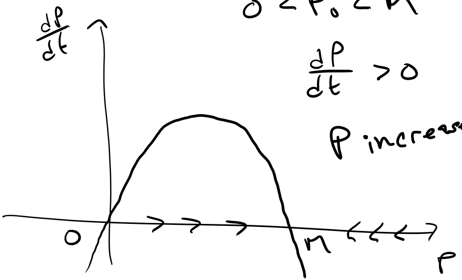
$\frac{dP}{dt} > 0$

P increases

$M < P_0$

$\frac{dP}{dt} < 0$

P decreases



Logistic Modeling

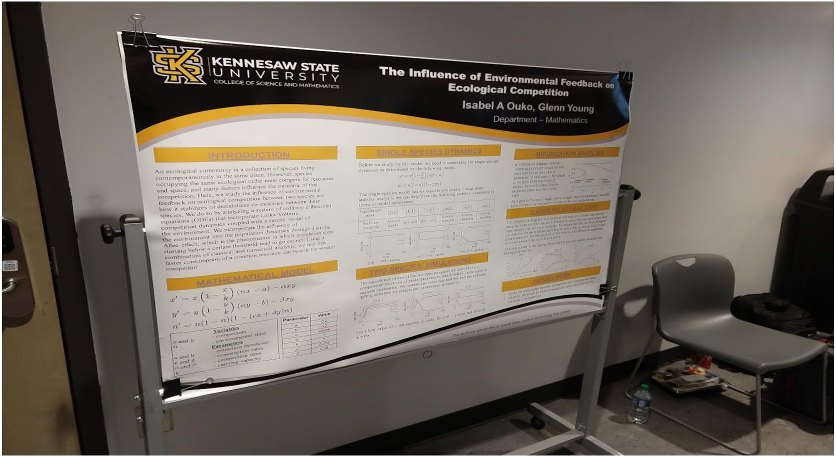
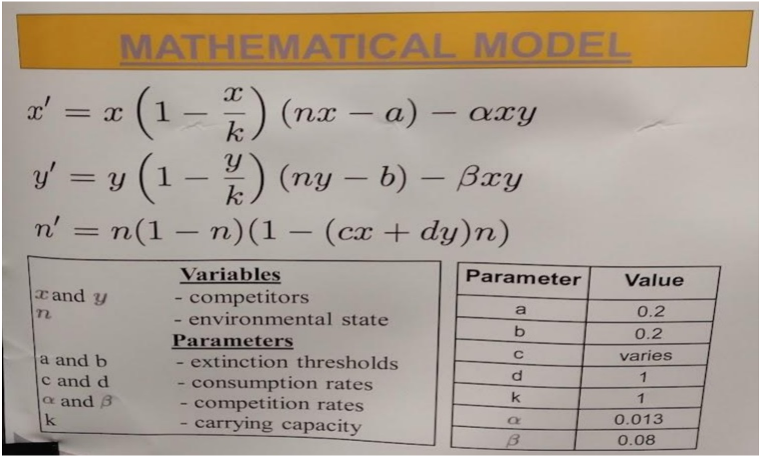


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling



MATHEMATICAL MODEL

$$x' = x\left(1 - \frac{x}{k}\right)(nx - a) - \alpha xy$$

$$y' = y\left(1 - \frac{y}{k}\right)(ny - b) - \beta xy$$

$$n' = n(1 - n)(1 - (cx + dy)n)$$

	Variables
x and y	- competitors
n	- environmental state
	Parameters
a and b	- extinction thresholds
c and d	- consumption rates
α and β	- competition rates
k	- carrying capacity

Parameter	Value
a	0.2
b	0.2
c	varies
d	1
k	1
α	0.013
β	0.08

Figure: The species equations include an extended logistic term with threshold and competition.