September 12 Math 3260 sec. 51 Fall 2025

Chapter 2 Systems of Linear Equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

2.2.1 Gaussian Elimination

We're interested in solving such a system using **Gaussian** elimination. Recall that this involved three operations that result in an equivalent system:

- Multiply an equation by a nonzero scalar k. $kE_i \to E_i$ (scale)
- Interchange the position of any two equations. $E_i \leftrightarrow E_j$ (swap)
- Replace an equation with the sum of itself and a multiple of any other equation. $kE_i + E_j \to E_j$ (replace)

Example

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 5x_3 & = & 6 \\ 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \end{array}$$

$-3E_1 + E_2 \to E_2$

$-2E_1 + E_3 \to E_3$

$$\begin{array}{l} x_1 - x_2 - 5x_3 = 6 \\ \quad 4x_2 + 8x_3 = -8 \\ \quad x_2 + 2x_3 = -2 \end{array}$$

$\frac{1}{4} E_2 \to E_2$

$-3x_1 + 3x_2 + 15x_3 = -18$

$3x_1 + x_2 - 7x_3 = 10$

$-2x_1 + 2x_2 + 10x_3 = -12$

$2x_1 - x_2 - 8x_3 = 10$

$$x_1 - x_2 - 5x_3 = 6$$

$$x_2 + 2x_3 = -2$$

$$x_2 + 2x_3 = -2$$

$$-E_2 + E_3 \to E_3$$

$$x_1 - x_2 - 5x_3 = 6$$

$$x_2 + 2x_3 = -2$$

$$0 = 0$$

Back substitution

$$x_2 = -2 - 2x_3$$

x_3 can be any real number.

$$x_1 = 6 + x_2 + 5x_3$$
$$= 6 + (-2 - 2x_3) + 5x_3$$
$$= 4 + 3x_3$$

A parametric description of the solution

Let $x_3 = t$

$$x_1 = 4 + 3t$$
$$x_2 = -2 - 2t \quad , \quad t \in \mathbb{R}$$
$$x_3 = t$$

Converting to vector parametric

$$\vec{x} = \langle x_1, x_2, x_3 \rangle$$

$$\vec{x} = \langle 4+3t, -2-2t, t \rangle$$

$$= \langle 4, -2, 0 \rangle + \langle 3t, -2t, t \rangle$$

$$\vec{x} = \langle 4, -2, 0 \rangle + t\langle 3, -2, 1 \rangle$$

Other Solution Cases

An example of the no solution case: The system

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

Inconsistent systems always give rise to an equation that is false.

$$0 = \text{something nonzero}$$

2.3 Matrices

Matrix

A **matrix** (plural *matrices*) is a rectangular array of numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Each number, a_{ij}, is called an **entry** or an **element** of the matrix. If the matrix has m rows and n columns, we say that the **size** or **dimension** of the matrix is "m by n" and write $m \times n$.

$$a_{ij}$$

row index ↗ ↖ column index

Coefficient & Augmented Matrices

Given a system of linear equations with m equations and n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

the **coefficient matrix** for the system is the $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Coefficient & Augmented Matrices

Given a system of linear equations with m equations and n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

the **augmented matrix** for the system is the $m \times n + 1$ matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Example

Write the coefficient and augmented matrices for each system.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Coef. matrix is 2×4

$$\begin{bmatrix} 2 & 1 & -3 & 1 \\ -1 & 3 & 4 & -2 \end{bmatrix}$$

The augmented matrix is 2×5

$$\left[\begin{array}{cccc|c} 2 & 1 & -3 & 1 & -3 \\ -1 & 3 & 4 & -2 & 8 \end{array}\right].$$

Example

Write the coefficient and augmented matrices for each system.

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & + & 3x_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

Coef matrix 3×3

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}.$$

Augmented matrix is 3×4

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right].$$

Elementary Row Operations

We will use matrices to perform elimination without involving the symbols and variable names. We have three operations we can perform on a matrix. We'll use the notation

$$R_i$$

to refer to the i^{th} row of the matrix.

Elementary Row Operations

- Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.
- Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.
- Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalence

Definition: We will say that two matrices are **row equivalent** if one matrix can be obtained from the other by performing some sequence of elementary row operations.

Remark: So every time we do a row operation, the result is row equivalent matrix.

Theorem:

If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.