The Principle of Superposition

Says that if we have some solutions, say $y_1(x)$, $y_2(x)$, and $y_3(x)$ of a linear homogeneous equation, then every function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$$

is also a solution of that linear, homogeneous equation.

The expression

$$c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$$

is called a **linear combination** of the functions $y_1(x)$, $y_2(x)$, and $y_3(x)$.

We needed a criteria to distinguish functions or characterize their relationship to one another.

Linear Dependence or Independence

Suppose we have a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on some interval I. We can consider the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

Note that it's always possible to pick c's to make this true (e.g. you can always set all the c values to zero). We'll say that the functions are

- **Linearly Dependent** if the equation can be made true **with at least one c being nonzero.**

- **Linearly Independent** if the only way the equation can be true is if all the c's must be zero.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Can we find c_1, c_2, c_2 such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all real x ?

We might notice that

$$f_3(x) = \frac{1}{4} f_2(x) - f_1(x)$$

$$x - x^2 = \frac{1}{4}(4x) - x^2 \quad \checkmark$$

We can arrange the equation moving

every thing to the left

$$f_1(x) - \frac{1}{4} f_2(x) + f_3(x) = 0$$

This has the form $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

where $c_1 = 1$, $c_2 = \frac{-1}{4}$, and $c_3 = 1$,

These are not all zero, so the functions are <u>linearly dependent</u>.

Linear Dependence Relation

An equation with at least one c nonzero, such as

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

from this last example is called a **linear dependence relation** for the functions $\{f_1, f_2, f_3\}$.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$f_1'(x) = \cos x$

$f_2'(x) = -\sin x$

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -1$$

$$W(\sin x, \cos x)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\left(-8-0\right) - 4x\left(-4x - 2(1-2x)\right) + (x - x^2)\left(-8\right)$$

$$= -8x^2 - 4x\left(-2\right) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W\left(x^2, 4x, x - x^2\right)(x) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

Compute W. If $W=0$ dependent

If not then independent.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

Let's use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x}$$

$$= -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x}$$

Since this is not zero, the functions are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We need to show that we have two, linearly independent solutions. The ODE is 2nd order. We have two, y_1 and y_2.

Let's verify that they are solutions.

$y_1 = x^2$ substitute

$y_1' = 2x$ $\qquad x^2y_1'' - 4xy_1' + 6y_1 \stackrel{?}{=} 0$

$y_1'' = 2$ $\qquad x^2(2) - 4x(2x) + 6x^2 \stackrel{?}{=} 0$

$2x^2 - 8x^2 + 6x^2 = 0$

y_1 is a solution $\qquad 0 = 0$

$$y_2 = x^3 \qquad x^2 y_2'' - 4x y_2' + 6y_2 \overset{?}{=} 0$$

$$y_2' = 3x^2 \qquad x^2(6x) - 4x.(3x^2) + 6x^3 \overset{?}{=} 0$$

$$y_2'' = 6x \qquad 6x^3 - 12x^3 + 6x^3 \overset{?}{=} 0$$

$$0 = 0$$

y_2 is also a solution

Compute the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4 = x^4$$

$$W(y_1, y_2)(x) = x^4 \quad \text{not zero}.$$

They are linearly independent.

We have a fundamental solution set, and the general solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 x^2 + c_2 x^3.$$