September 13 Math 2306 sec. 52 Spring 2023

Section 5: First Order Equations: Models and Applications

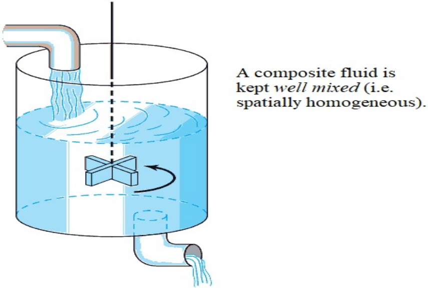

Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

Some Notation

In addition to the amount of salt, $A(t)$, at time t we have several variables or parameters. Let

- r_i be the rate at which fluid enters the tank (rate in),
- r_o be the rate at which fluid leaves the tank (rate out),
- c_i be the concentration of substance (salt) in the in-flowing fluid (concentration in),
- c_o be the concentration of substance (salt) in the out-flowing fluid (concentration out),
- $V(t)$ be the total volume of fluid in the tank at time t,
- V_0 be the volume of fluid in the tank at time $t = 0$, i.e., $V_0 = V(0)$

A Classic Mixing Problem Illustrated

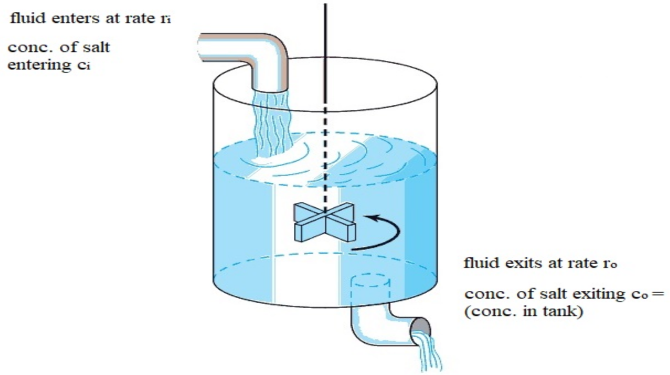

Figure: Values for c_i, r_i, and r_o are given in the problem statement. The well mixed assumption means that c_o will match the concentration in the tank.

This means that c_o is **NOT constant**! It depends on time through both A and V.

Building an Equation

What is the rate of change of the mass of the salt?

$$\frac{dA}{dt} = \left(\begin{array}{c}\textit{input rate}\\ \textit{of salt}\end{array}\right) - \left(\begin{array}{c}\textit{output rate}\\ \textit{of salt}\end{array}\right)$$

where

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

The parameters r_i, c_i, and r_o are part of the problem statement. We must determine c_o.

Building an Equation

By the well mixed solution assumption, the concentration of salt in the out-flowing fluid matches the concentration in the tank. That is,

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

Note that the volume

$$V(t) = \textit{initial volume} + \textit{rate in} \times \textit{time} - \textit{rate out} \times \textit{time}.$$

If $r_i = r_o$, then $V(t) = V(0)$ a constant.

Pulling this together, the amount A satisfies the first order linear ODE

$$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.}$$

$\frac{A}{V} = c_o$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

The model is $\frac{dA}{dt} = r_i c_i - r_o c_o = r_i c_i - r_o \frac{A}{V}$

we need V, r_i, c_i, and r_o.

$V(0) = 500$ gal, $c_i = 2 \frac{\text{lb}}{\text{gal}}$, $r_i = 5 \frac{\text{gal}}{\text{min}}$

$r_o = 5 \frac{\text{gal}}{\text{min}}$. The volume

$V(t) = V(0) + (r_i - r_o)t = 500 \text{ gal} + \left(5 \frac{\text{gal}}{\text{min}} - 5 \frac{\text{gal}}{\text{min}}\right) t \text{ min}$

$V(t) = 500$ gal.

The initial amount of salt $A(0)=0$ since it starts w/ pure water.

$$\frac{dA}{dt} = 5\,\frac{\text{gal}}{\text{min}} \cdot 2\,\frac{\text{lb}}{\text{gal}} - 5\,\frac{\text{gal}}{\text{min}}\,\frac{A\ \text{lb}}{500\ \text{gal}}, \quad A(0)=0$$

$$\frac{dA}{dt} = 10 - \frac{1}{100}A, \quad A(0)=0.$$

This is both linear and separable.

$$\frac{dA}{dt} + \frac{1}{100}A = 10$$

$$P(t) = \frac{1}{100}, \quad \mu = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}A\right) = 10\,e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t}A\right)dt = \int 10e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t} A = 10(100) e^{\frac{1}{100}t} + C$$

$$A = \frac{1000 e^{\frac{1}{100}t} + C}{e^{\frac{1}{100}t}} = 1000 + C e^{-\frac{1}{100}t}$$

Apply $A(0) = 0$

$$A(0) = 1000 + C e^{0} = 0 \Rightarrow C = -1000$$

The amount of salt in the tank @ time t minutes is

$$\boxed{A(t) = 1000 - 1000 e^{-\frac{1}{100}t}}$$

The concentration of salt in the tank @ time t is

$$\text{Conc.} = \frac{A(t)}{V(t)} \ \frac{\text{lb}}{\text{gal}}$$

When $t=5$, $\text{Conc.} = \frac{A(5)\ \text{lb}}{500\ \text{gal}} = \frac{1000 - 1000e^{\frac{-1}{100}(5)}}{500} \ \frac{\text{lb}}{\text{gal}}$

$$\approx 0.1 \ \frac{\text{lb}}{\text{gal}}$$

We can ask what the concentration of salt in the tank is expected to be after a long time.

$$\text{Conc.} = \frac{A(t)}{V(t)} = \frac{1000 - 1000e^{\frac{-1}{100}t}}{500} \ \frac{\text{lb}}{\text{gal}}$$

As $t \to \infty$ $\text{Conc.} \to \frac{1000 - 0}{500} = 2 \ \frac{\text{lb}}{\text{gal}}$

A Nonlinear Modeling Problem

The last model we will consider is a nonlinear population model. It can account for reproduction and environmental limitations. Let's consider it through an example.

A population $P(t)$ of tilapia changes at a $\underbrace{\text{rate}}^{\frac{dP}{dt}}$ jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

Current pop = P

Difference between carrying capacity and the population is $M-P$

$$\frac{dP}{dt} \propto P(M-P) \implies \frac{dP}{dt} = kP(M-P) \quad \text{for some constant } k.$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

Logistic Growth Model

The equation $\dfrac{dP}{dt} = kP(M-P)$, where $k, M > 0$ is called a **logistic growth equation**.

Suppose the intial population $P(0) = P_0$. Solve the resulting initial value problem. Show that if $P_0 > 0$, the population tends to the carrying capacity M.

The ODE is separable and Bernoulli

Bernoulli

$$\frac{dy}{dt} + Q(t)y = f(t)y^n$$

$$\frac{dP}{dt} = kMP - kP^2$$

$$\frac{dP}{dt} - kMP = -kP^2$$

$n = 2$, $Q(t) = -kM$, $f(t) = -k$, $1-n = 1-2 = -1$

Logistic Growth: $P'(t) = kP(M-P)\; P(0) = P_0$

Let $u = P^{1-n} = P^{1-2} = P^{-1}$, $u = \frac{1}{P}$ so $P = \frac{1}{u}$

u solves $\frac{du}{dt} + (1-n)Q(t)u = (1-n)f(t)$

$$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$Q_1(t) = kM$, $\mu = e^{\int Q_1(t)\,dt} = e^{\int kM\,dt} = e^{kMt}$

$$\frac{d}{dt}\left(e^{kMt}u\right) = k e^{kMt}$$

$$\int \frac{d}{dt}\left(e^{kMt}u\right)dt = \int k e^{kMt}\,dt$$

$$e^{kMt} u = k \frac{1}{kM} e^{kMt} + C$$

$$u = \frac{\frac{1}{M} e^{kMt} + C}{e^{kMt}} = \frac{1}{M} + C e^{-kMt} \cdot \frac{M}{M}$$

$$u = \frac{1 + CMe^{-kMt}}{M}, \quad \text{let} \quad C_1 = CM$$

$$u = \frac{1 + C_1 e^{-kMt}}{M} \qquad P = \frac{1}{u}$$

so $$P(t) = \frac{M}{1 + C_1 e^{-kMt}}$$

Apply $P(0) = P_0$ to find C_1

$$P(0) = \frac{M}{1+C_1 e^0} = P_0 \Rightarrow \frac{M}{1+C_1} = P_0$$

$$M = (1+C_1)P_0 = P_0 + C_1 P_0 \Rightarrow C_1 P_0 = M - P_0$$

$$C_1 = \frac{M-P_0}{P_0}$$

The population

$$P(t) = \frac{M}{1+\left(\frac{M-P_0}{P_0}\right)e^{-kMt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0+(M-P_0)e^{-kMt}} = \frac{MP_0}{P_0+0} = M$$

Long Time Solution of Logistic Equation

$$\frac{dP}{dt} = kP(M-P) = -kP^2 + kMP.$$

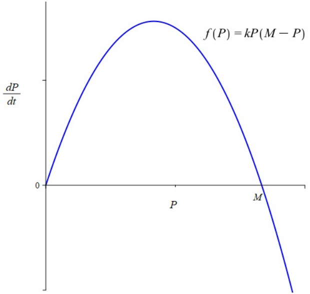

Figure: Plot of P versus $\frac{dP}{dt}$. Note that $\frac{dP}{dt} > 0$ if $0 < P < M$ and $\frac{dP}{dt} < 0$ if $P > M$.

Expected Long Time Solutions

Suppose we modify the logistic equation based on the assumption that the fish will only breed successfully if the population is above some minimum threshhold N where $0 < N < M$. The new model is

$$\frac{dP}{dt} = kP(M-P)(P-N).$$

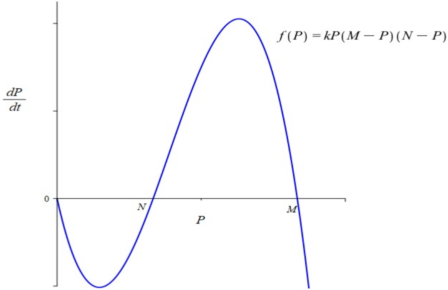

Figure: Plot of P versus $\frac{dP}{dt}$ for the modified model. There are two long-time scenarios, extinction and achieving carrying capacity.

Models Derived in this Section

We have several models involving first order ODEs.

Exponential Growth/Decay $$\boxed{\frac{dP}{dt} = kP}$$

RC-Series Circuit $$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

LR-Series Circuit $$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

Classical Mixing $$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}}$$

Logistic Growth $$\boxed{\frac{dP}{dt} = kP(M - P)}$$