September 14 Math 2306 sec. 51 Fall 2022

Section 5: First Order Equations Models and Applications

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

We solved this equation subject to the initial condition $P(0) = P_0$ and arrived at

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}.$$

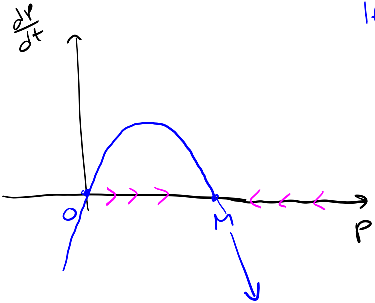
If $P_0 = 0$, then $P(t) = 0$ for all t. If $P_0 > 0$, then

$$\lim_{t\to\infty} P(t) = M.$$

Logistic Growth Model

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

$$\frac{dP}{dt} = f(P) = kP(M-P) = kMP - kP^2$$



If $0 < P < M$

$\frac{dP}{dt} > 0$ P increases

If $P > M$

$\frac{dP}{dt} < 0$

P decreases

Logistic Modeling

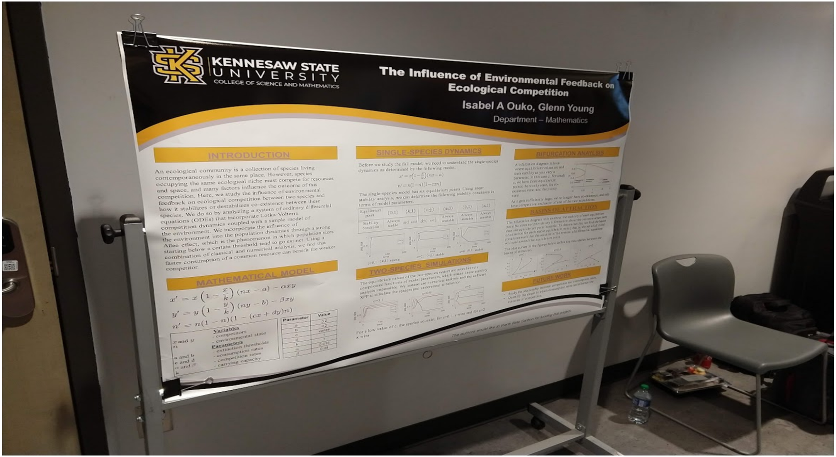


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling

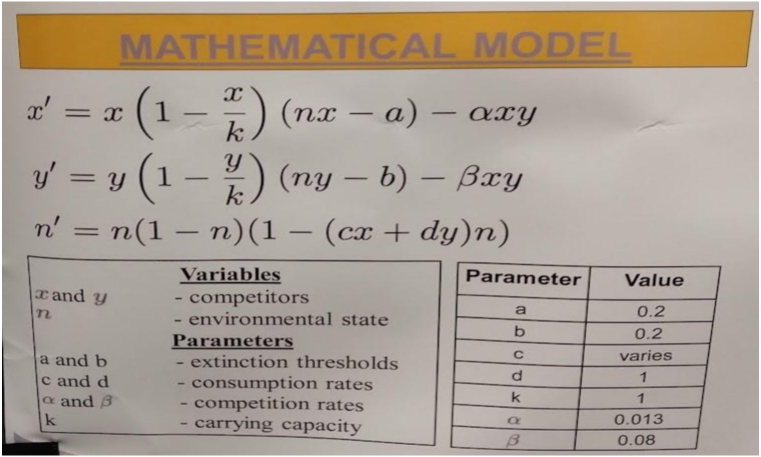


Figure: The species equations include an extended logistic term with threshold and competition.

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

Theorem: If $a_0,\ldots,a_n$ and g are continuous on an interval I, $a_n(x)\neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0,\ldots,y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

The Principle of Superposition (homogeneous ode)

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Assume a_i are continuous and $a_n(x)\neq 0$ for all x in I.

Theorem: If y_1, $y_2,\ldots,y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1,\ldots,c_k$.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Consider the equation

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all } x$$

$$c_1 \sin x + c_2 \cos x = 0$$

Assume this is true.

The equation is true when $x = 0$.

so $\quad c_1 \sin 0 + c_2 \cos 0 = 0$

$$\Rightarrow \quad c_1(0) + c_2(1) = 0 \quad \Rightarrow \quad c_2 = 0$$

The equation is also true if $x = \pi/2$

$$c_1 \sin\frac{\pi}{2} + 0 \cdot \cos \pi/2 = 0$$

$$c_1(1) = 0 \implies c_1 = 0$$

$\implies c_1 = c_2 = 0$ is the only way the equation is true.