September 14 Math 2306 sec. 52 Fall 2022

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

The Principle of Superposition (homogeneous ode)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Theorem: If y_1, $y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x.

That is, $c_1 \sin x + c_2 \cos x = 0$. We'll show that c_1 and c_2 must be zero.

The equation must be true when $x = 0$.

So $c_1 \sin 0 + c_2 \cos 0 = 0$

$c_1(0) + c_2(1) = 0 \implies c_2 = 0$

The equation holds when $x = \pi/2$

$$C_1 \sin \pi/2 + 0 \cos \pi/2 = 0$$

This says $C_1(1) = 0$ i.e. $C_1 = 0$

Since $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x

is only true if $c_1 = 0$ and $c_2 = 0$,

$f_1(x)$ and $f_2(x)$ are linearly independent.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Notice that

$$f_3(x) = \frac{1}{4} f_2(x) - f_1(x)$$

Check:

$$x - x^2 \stackrel{?}{=} \frac{1}{4}(4x) - x^2$$

$$x - x^2 = x - x^2 \quad \checkmark$$

We can rearrange the equation, move f_1

and f_2 to the left

$$f_1(x) - \frac{1}{4} f_2(x) + f_3(x) = 0$$

This is $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

with $c_1 = 1$, $c_2 = \frac{-1}{4}$ and $c_3 = 1$

This is a linearly dependent set.