September 15 Math 2306 sec. 51 Fall 2021

Section 6: Linear Equations Theory and Terminology

We were talking about the basics of linear homogeneous ODEs.

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

The General Solution: Let $y_1, y_2, \dots, y_n$ be any fundamental solution set for this ODE. The general solution to this homogeneous equation is

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x),$$

where $c_1, \dots, c_n$ are arbitrary constants.

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{this is } y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_1(x)+g_2(x) \quad (1)$$

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_2(x),$$

then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution for the nonhomogeneous equation (1).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

We'll sub y_{p_1} in.

$$y_{p_1} = 6, \quad y_{p_1}' = 0, \quad y_{p_1}'' = 0$$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

an identity

y_{p_1} does solve the ODE.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

We sub y_{p_2} in. $y_{p_2} = -7x$, $y_{p_2}' = -7$, $y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6 y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x \quad \text{an identity.}$$

So y_{p_2} does solve this ODE.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y_{p_1} = 6$

$y_{p_2} = -7x$

We know that $y = y_c + y_p$

$y_c = C_1 x^2 + C_2 x^3$. and $y_p = y_{p_1} + y_{p_2}$

$= 6 - 7x$

The general solution is

$$\boxed{y = C_1 x^2 + C_2 x^3 + 6 - 7x}$$

Solve the IVP

IC

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

We have the general solution

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$

$$y' = 2c_1x + 3c_2x^2 - 7$$

Apply the IC

$$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0 \Rightarrow c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = -5$$

$$2c_1 + 3c_2 - 7 = -5 \Rightarrow 2c_1 + 3c_2 = 2$$

We need to solve the system

$$c_1 + c_2 = 1$$ ← mult by $\cdot 2$ add to other equation

$$2c_1 + 3c_2 = 2$$

$$-2c_1 - 2c_2 = -2$$

$$c_2 = 0$$

$$c_1 = 1 - c_2 = 1 - 0 \implies c_1 = 1$$

The solution to the IVP is

$$y = 1x^2 + 0x^3 + 6 - 7x$$

i.e. $\boxed{y = x^2 + 6 - 7x}$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Assume $y_2 = u y_1$ * u can't be constant because y_1 + y_2 are lin. independent.

We'll sub y_2 into the ODE. Find y_2' + y_2''

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)(u'y_1 + uy_1') + Q(x)uy_1 = 0$$

Collect u, u' and u''

$$u''y_1 + (2y_1' + P(x)y_1)u' + \underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{0}u = 0$$

since y_1 solves the ODE

The ODE for u is

$$y_1u'' + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$, we have a 1st order ODE for w.

$$w' + \left(\frac{2y_1'}{y_1} + P(x)\right) w = 0$$

This is linear and separable.

$$w' = -\left(\frac{2y_1'}{y_1} + P(x)\right) w$$

$$\frac{dw}{w} = -\left(\frac{2y_1'}{y_1} + P(x)\right) dx$$

$$\int \frac{dw}{w} = -\int \frac{2\,dy_1}{y_1} - \int P(x)\,dx$$

assume $W > 0$

$$\ln W = -2\ln|y_1| - \int P(x)\,dx$$

exponentiate

$$W = e^{-2\ln|y_1| - \int P(x)dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int P(x)\,dx}$$

$$W = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

Since $W = u'$, $u = \int W\,dx$

$$u = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx .$$

and $\quad y_2 = u y_1$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$