September 15 Math 2306 sec. 52 Fall 2021

Section 6: Linear Equations Theory and Terminology

We were talking about the basics of linear homogeneous ODEs.

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

The General Solution: Let $y_1, y_2, \dots, y_n$ be any fundamental solution set for this ODE. The general solution to this homogeneous equation is

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x),$$

where $c_1, \dots, c_n$ are arbitrary constants.

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_1(x)+g_2(x) \quad (1)$$

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_2(x),$$

then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution for the nonhomogeneous equation (1).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

we'll sub y_{p_1} in. $y_{p_1} = 6$, $y_{p_1}' = 0$, $y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} \overset{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \overset{?}{=} 36$$

$$36 = 36$$

this is an identity

y_{p_1} does solve the ODE.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

We sub y_{p_2} in. $y_{p_2} = -7x$, $y_{p_2}' = -7$, $y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x$$

an identity.

So y_{p_2} solve the ODE.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y_{p_1} = 6 \qquad y_{p_2} = -7x$

We know $y = y_c + y_p$

$y_c = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^3$ and $y_p = y_{p_1} + y_{p_2}$

$= 6 - 7x$

The general solution

$$\boxed{y = c_1x^2 + c_2x^3 + 6 - 7x}$$

Solve the IVP

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$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution (from the last example)

is $\quad y = c_1x^2 + c_2x^3 + 6 - 7x$

Apply the I.C.

$$y' = 2c_1x + 3c_2x^2 - 7$$

$$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0 \implies c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = -5$$

$$2c_1 + 3c_2 - 7 = -5 \implies 2c_1 + 3c_2 = 2$$

We solve the system

$$c_1 + c_2 = 1$$

$$2c_1 + 3c_2 = 2$$

mult first eqn by -2 and add

$$-2c_1 - 2c_2 = -2$$

$$c_2 = 0 \qquad c_1 = 1 - c_2 = 1 - 0 = 1$$

The solution to the IVP

$$y = 1x^2 + 0x^3 + 6 - 7x$$

$$\boxed{y = x^2 + 6 - 7x}$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x) y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Note u can't be a constant function.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

We assume $y_2 = uy_1$. We'll sub this into the ODE.

$$y_2 = uy_1$$
$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$
$$= u''y_1 + 2u'y_1' + uy_1''$$

Sub $y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0$

$$u'' y_1 + 2u' y_1' + u y_1'' + P(x)\left(u' y_1 + u y_1'\right) + Q(x)\, u y_1 = 0$$

Collect $u,\ u',\ u''$

$$y_1 u'' + \left(2y_1' + P(x)\, y_1\right) u' + \underbrace{\left(y_1'' + P(x)\, y_1' + Q(x)\, y_1\right)}_{0 \text{ '' } y_1 \text{ is a solution}} u = 0$$

So u solves

$$y_1 u'' + \left(2y_1' + P(x)\, y_1\right) u' = 0$$

Let $w = u'$, then $w' = u''$

and w solves

$$w' + \left(\frac{2y_1'}{y_1} + p(x)\right)w = 0$$

A 1^{st} order linear and separable ODE,

$$w' = -\left(\frac{2y_1'}{y_1} + p(x)\right)w$$

$$\frac{dw}{w} = -\left(\frac{2y_1'}{y_1} + p(x)\right)dx$$

$$\int \frac{dw}{w} = -\int \frac{2\,dy_1}{y_1} - \int p(x)\,dx$$

assuming $w>0$

$$\ln w = -2\ln|y_1| - \int p(x)\,dx$$

$$W = e^{-2\ln|y_1| - \int P(x)\,dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int p(x)\,dx}$$

$$W = y_1^{-2} e^{-\int p(x)\,dx}$$

$$W = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

$$W = u' \implies u = \int w\,dx$$

So $u = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

and $y_2 = u y_1$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$