September 15 Math 2306 sec. 52 Spring 2023

Section 5: First Order Equations: Models and Applications

Logistic Growth Model

The equation $\frac{dP}{dt} = kP(M-P)$, where $k, M > 0$ is called a **logistic growth equation**.

When coupled with the initial condition $P(0) = P_0$, the solution to the resulting IVP is

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}.$$

We saw that if $P_0 = 0$, then $P(t) = 0$ for all $t > 0$, and if $P_0 > 0$, then $P(t) \to M$ as $t \to \infty$.

Long Time Solution of Logistic Equation

$$\frac{dP}{dt} = kP(M - P) = -kP^2 + kMP.$$

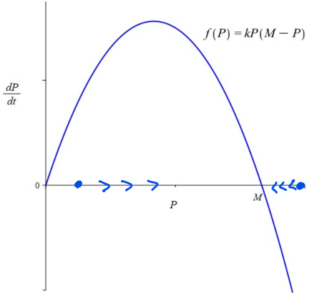


Figure: Plot of P versus $\frac{dP}{dt}$. Note that $\frac{dP}{dt} > 0$ if $0 < P < M$ and $\frac{dP}{dt} < 0$ if $P > M$.

Using the Logistic Growth Model

In practice, the basic logistic growth model is often used as a starting point, and additional considerations affecting a population can be factored in. Some common extensions include

- Constant rate harvesting (e.g., fishing): $\frac{dP}{dt} = kP(M-P) - H$,
- Popluation dependent harvesting: $\frac{dP}{dt} = kP(M-P) - HP$,
- Restocking: $\frac{dP}{dt} = kP(M-P) + R$
- Periodic harvesting/restocking: $\frac{dP}{dt} = kP(M-P) + r\sin(\omega t)$
- Threshhold dependent breeding: $\frac{dP}{dt} = kP(M-P)(P-N)$

Expected Long Time Solutions

Suppose we modify the logistic equation based on the assumption that the fish will only breed successfully if the population is above some minimum threshhold N where $0 < N < M$. The new model is

$$\frac{dP}{dt} = kP(M-P)(P-N).$$

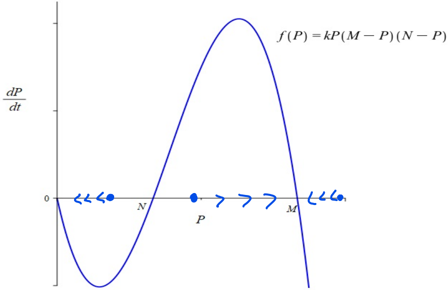


Figure: Plot of P versus $\frac{dP}{dt}$ for the modified model. There are two long-time scenarios, extinction and achieving carrying capacity.

Logistic Modeling

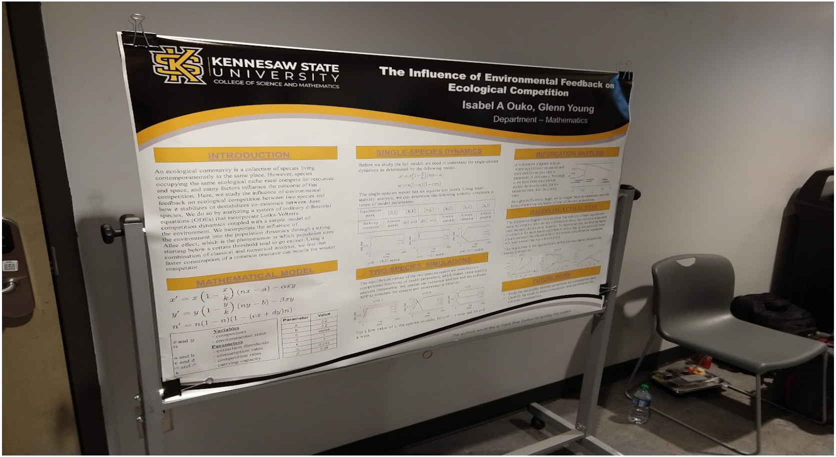
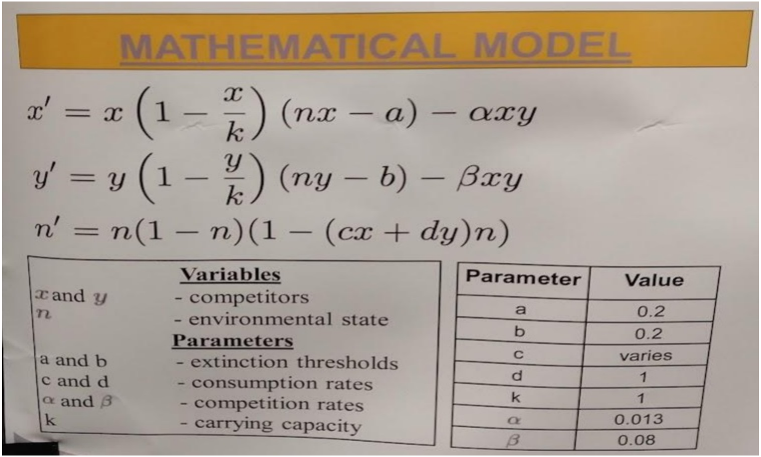


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling



MATHEMATICAL MODEL

$$x' = x\left(1 - \frac{x}{k}\right)(nx - a) - \alpha xy$$

$$y' = y\left(1 - \frac{y}{k}\right)(ny - b) - \beta xy$$

$$n' = n(1 - n)(1 - (cx + dy)n)$$

	Variables
x and y	- competitors
n	- environmental state
	Parameters
a and b	- extinction thresholds
c and d	- consumption rates
α and β	- competition rates
k	- carrying capacity

Parameter	Value
a	0.2
b	0.2
c	varies
d	1
k	1
α	0.013
β	0.08

Figure: The species equations include an extended logistic term with threshold and competition.

Models Derived in this Section

We have several models involving first order ODEs.

Exponential Growth/Decay

$$\frac{dP}{dt} = kP$$

RC-Series Circuit

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

LR-Series Circuit

$$L\frac{di}{dt} + Ri = E(t)$$

Classical Mixing

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}$$

Logistic Growth

$$\frac{dP}{dt} = kP(M - P)$$