September 15 Math 3260 sec. 51 Fall 2025

2.3 Matrices

We defined a matrix as a rectangular arraw of numbers. Given a system of linear equations with m equations and n variables, we defined the coefficient and the augmented matrix.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n \text{ coefficient matrix}} \quad \underbrace{\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]}_{m\times(n+1) \text{ augmented matrix}}.$$

Elementary Row Operations

We will use matrices to perform elimination without involving the symbols and variable names. We have three operations we can perform on a matrix. We'll use the notation

$$R_i$$

to refer to the i^{th} row of the matrix.

Elementary Row Operations

- Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.
- Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.
- Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalence

Definition: We will say that two matrices are **row equivalent** if one matrix can be obtained from the other by performing some sequence of elementary row operations.

Remark: So every time we do a row operation, the result is row equivalent matrix.

Theorem:

If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Gaussian Elimination & Augmented Matrix Structure

A Consistent System (one solution)

From

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \left[\begin{array}{rrr|r} 2 & 0 & 1 & 7 \\ 1 & 2 & -1 & -4 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Note a sort of *triangular/trapezoidal* shape. Each row has a nonzero term on the left of the bar between the coefficient and augment columns.

Gaussian Elimination & Augmented Matrix Structure

A Consistent System (infinitely many solutions)

From

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 5x_3 & = & 6 \\ 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -5 & 6 \\ 3 & 1 & -7 & 10 \\ 2 & -1 & -8 & 10 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 5x_3 & = & 6 \\ & & x_2 & + & 2x_3 & = & -2 \\ & & & & 0 & = & 0 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & -1 & -5 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This also has a triangular/trapezoidal structure. **There are fewer nonzero rows than there are columns to the left of the bar.**

Caveat: The tell is not *a row of zeros*, it's fewer nonzero rows than columns to the left of the bar.

Gaussian Elimination & Augmented Matrix Structure

An Inconsistent System

From

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 4 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 4 & 0 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & 4 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Again, there is a triangle/trapezoid shape. A row with the only nonzero entry on the right side of the bar, $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$, alerts us to a false statement.

When reading the row of a matrix, from left to right, we will call the first nonzero entry a **leading entry**.

Definition: Echelon Forms

We will say that a matrix is in **row echelon form** (ref) if it satisfies the properties that

1. any row whose entries are all zeros is below all rows that contain a leading entry, and
2. the leading entry in every row is to the right of the leading entries in every row above it.

We will say that a matrix is in **reduced row echelon form** (rref) if, in addition to being in row echelon form,

3. the leading entry in each row is a 1 (called a “leading one”), and
4. each leading one is the only nonzero entry in its column.

Example

Classify each matrix as a **row echelon form (ref)**, a **reduced row echelon form (rref)**, or not an echelon form.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A is a ref. not an rref

B is not an echelon form The leading entry in row 3 is not to the right of the leading entry in row 2.

C is an ref. C is an rref.

Example

Classify each matrix as a **row echelon form (ref)**, a **reduced row echelon form (rref)**, or not an echelon form.

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

D is an ref.

D is not an rref.
The leading 1 in
Column 2 has a
non zero entry
in it's column

E is ref; E
is rref.

Row Reduction Algorithm

We will perform row operations to reduce A to an ref, and then to an rref. This is a two stage process that we can approach methodically.

- **Forward:** We work from left to right, top down to obtain an ref.
- **Backward:** From an ref, we work from right to left, bottom up to clear nonzero entries above the leading entries.

$$A = \begin{bmatrix} 3 & 2 & 1 & 6 & 0 \\ 4 & 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & 3 & -2 \\ 2 & 1 & 1 & 3 & 2 \end{bmatrix}$$

First goal is to turn the first column into $\begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix}$

something not zero

$kR_1 + R_i \to R_i$

$R_1 \leftrightarrow R_3$

putting a 1 in the top entry for convenience

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 4 & 2 & 2 & 0 & -2 \\ 3 & 2 & 1 & 6 & 0 \\ 2 & 1 & 1 & 3 & 2 \end{bmatrix}$$

$-4R_1 + R_2 \to R_2$

$-3R_1 + R_3 \to R_3$

$-2R_1 + R_4 \to R_4$

-4	-4	0	-12	8
4	2	2	0	-2

-3	-3	0	-9	6
3	2	1	6	0

-2	-2	0	-6	4
2	1	1	3	2

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 0 & -2 & 2 & -12 & 6 \\ 0 & -1 & 1 & -3 & 6 \\ 0 & -1 & 1 & -3 & 6 \end{bmatrix}$$

$-\frac{1}{2} R_2 \to R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 6 & -3 \\ 0 & -1 & 1 & -3 & 6 \\ 0 & -1 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 + R_3 \to R_3$

$R_2 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 6 & -3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} \qquad -R_3 + R_4 \to R_4$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 6 & -3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

this is an ref

$$\tfrac{1}{3} R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 6 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} -6R_3 + R_2 \to R_2 \\ -3R_3 + R_1 \to R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & -1 & 0 & -9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad -R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & -1 & 0 & -9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an rref.

To recap, work from left to right to find leading terms and get zeros below them. This leads to an ref. From the ref, the locations of the leading 1s will be known. Work from right to left to get zeros above the leading entries and scale the leading entries as needed to be 1.

Note: A column that doesn't have a leading 1 in it will have whatever numbers it ends up with. Those can't be controlled, you just gotta let them be whatever they turn out to be.

Another note: Every matrix can be reduced to a row equivalent rref. It doesn't have to be an augmented matrix, and there's no such thing as a matrix that can't be reduced.

Uniqueness of an rref

Theorem:

A matrix A is row equivalent to exactly one reduced echelon form.

Remark: A matrix can be row equivalent to lots of different refs, but to only one RREF. So it makes sense to call it THE rref, and to write

$$\text{rref}(A).$$

Pivot Position & Pivot Column

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.