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Section 6: Linear Equations Theory and Terminology

Consider the second order, linear ODE

$$x^2y'' - xy' + y = 1.$$

It is easy to show that $y = x + 1$ is a solution.

$y = x+1$

$y' = 1$

$y'' = 0$

$x^2(0) - x(1) + (x+1) \stackrel{?}{=} 1$

$-x + x + 1 \stackrel{?}{=} 1$

$1 = 1$ ✓

$$x^2y'' - xy' + y = 1$$

Here are 10 more solutions to this ODE!

$$y = 1 \qquad y = x\ln x + 1$$

$$y = 3x - x\ln x + 1 \qquad y = 7x\ln x + 8x + 1$$

$$y = 1 - 4x\ln\sqrt{x} \qquad y = 5x\ln\left(\frac{1}{x}\right) + 1 - x$$

$$y = 16x + x\ln x + 1 \qquad y = 1 - x\ln x^3$$

$$y = 16x\ln x^2 + \frac{2}{7}x + 1 \qquad y = \frac{x}{3} + x\ln x^7 + 1$$

An IVP

Consider the IVP

$$x^2y'' - xy' + y = 1, \quad y(1) = 1, \quad y'(1) = -1$$

Not one of the eleven solutions that I showed solve this IVP!

This raises some questions.

- What do mean when we talk about **solving** an ODE or an IVP?
- How do we know when we're done **solving** an ODE?
- Is there something we would call **THE solution**?

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous.**

Theorem: Existence & Uniqueness

We have the following important theorem regarding the existence and uniqueness of solutions to the IVP

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem:

If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Some Goals

We want to know how to construct solutions, what solutions will look like, for linear ODEs. Some important terms will be

- complementary solution,
- particular solution,
- general solution

First, we will focus on **homogeneous** equations. We will consider the ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

In what follows, we will assume that $a_i(x)$ is continuous on some interval I and that $a_n(x) \neq 0$ for all x in I.

After we know what to expect for homogeneous equations, we'll come back to nonhomogeneous equations.

Recall

In the section 2 homework, there was this exercise

Problem 4. Consider the third order equation

$$y''' + 2y'' + y' = 0 \quad \text{for } -\infty < x < \infty.$$

(a) Verify that each of the functions $y_1 = e^{-x}$, $y_2 = xe^{-x}$ and $y_3 = 1$ solve this ODE.

(b) Show that the function $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ is a solution of the ODE for any constants c_1, c_2, c_3. (This is called the *general solution*.)

(c) Solve the IVP $\quad y''' + 2y'' + y' = 0, \quad y(0) = 6, \quad y'(0) = -3, \quad y''(0) = 4.$

(d) It is obvious that $Y = c_1 e^{-x} + c_2 xe^{-x}$ is a solution to the ODE $Y''' + 2Y'' + Y' = 0$ for any choice of constants c_1 and c_2. (Why is this now obvious?) Show that it is not possible to find constants c_1 and c_2 such that $Y = c_1 e^{-x} + c_2 xe^{-x}$ solves the IVP

$$Y''' + 2Y'' + Y' = 0 \qquad Y(0) = 1, \quad Y'(0) = 2, \quad Y''(0) = 3$$

This suggests that we have to be concerned with finding *all possible solutions* as opposed to just finding a solution when dealing with ODEs.

Superposition

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Theorem: The Principle of Superposition

If y_1, $y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

Remark 1: This result, known as the principle of superposition says that new solutions to the homogeneous equation can be constructed by multiplying solutions by constants and adding them together.

Remark 2: This is the principle of superposition for **homogeneous**, linear ODEs. We will state another principle for nonhomogeneous equations.

Corollaries

These two results follow directly from the principle of superposition for linear, homogeneous ODEs.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

This raies a couple of **Big Questions:**

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct? (The next definition will address this.)

Linear Dependence

Definition:

A set of functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants c_1, $c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Question: Can we make the sum equal to zero WITHOUT all the c's being zero?

YES $\Longrightarrow$ Linearly **Dependent.**

NO $\Longrightarrow$ Linearly **Independent.**

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Suppose there are constants c_1 and c_2

such that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x.

So $c_1 \sin x + c_2 \cos x = 0$ for all real x.

Since the equation is true for all x, it's true when $x=0$. When $x=0$, we get

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

i.e., $c_1(0) + c_2(1) = 0 \Rightarrow c_2 = 0.$

The equation is actually

$$c_1 \sin x = 0 \quad \text{for all real } x.$$

This holds when $x = \frac{\pi}{2}$, so that

$$c_1 \sin\left(\frac{\pi}{2}\right) = 0$$

$$c_1(1) = 0 \Rightarrow c_1 = 0$$

$c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x

only if $c_1 = c_2 = 0.$

So $f_1(x) = \sin x$, $f_2(x) = \cos x$

are linearly independent on

$(-\infty, \infty)$.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

We can consider the equation

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$ for all real x

we have

$$c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0$$

The x^2 will cancel if $c_1 = c_3$.

The x's will cancel if $c_2 = \frac{-1}{4} c_3$.

If we set $c_1 = 1$, then $c_3 = 1$ and $c_2 = \frac{-1}{4}$ will work.

$$1x^2 + \left(\frac{-1}{4}\right)(4x) + 1(x - x^2) = 0$$

$$x^2 - x + x - x^2 = 0$$

The set of coefficients $c_1 = 1$, $c_2 = \frac{-1}{4}$, $c_3 = 1$ are not all zero.

Hence $f_1(x) = x^2$, $f_2(x) = 4x$, $f_3(x) = x - x^2$ are linearly dependent on $(-\infty, \infty)$.

Linear Dependence Relation

Linear Dependence Relation

An equation with at least one c nonzero, such as

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

from this last example is called a **linear dependence relation** for the functions $\{f_1, f_2, f_3\}$.

With only two or three functions, we may be able to intuit linear dependence/independence. We have an object that will allow us to test for linear dependence under certain circumstances. This is the next topic.

Definition of Wronskian

Definition: Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Note that, in general, this Wronskian is a real valued function of the independent variable x. The notation allows us to indicate what functions the Wronskian depends on as well as the independent variable. We'll often shorten it to $W(x)$ or just W as long as it's clear from the context.

Determinant Formulas (2×2 and 3×3)

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

with 2 functions, the matrix will be 2x2.

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x\,(-\sin x) - \cos x\,(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$= -1$$

$$W(f_1, f_2)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ the matrix will be 3x3.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8-0) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(-8)$$

$$-4x - 2 + 4x$$

$$= -8x^2 - 4x(-2) - 8(x - x^2)$$

$$= -8x^2 + 8x - 8x + 8x^2 = 0$$

$W(f_1, f_2, f_3)(x) = 0$ for $f_1(x) = x^2$

$f_2(x) = 4x$

$f_3(x) = x - x^2$

Theorem (a test for linear independence)

Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that

$$W(f_1, f_2, \ldots, f_n)(x_0) \neq 0,$$

then the functions are **linearly independent** on I.

Remark: For the sorts of functions we're interested in, we can use this as a test:

$$W = 0 \implies \text{dependent} \quad \text{or} \quad W \neq 0 \implies \text{independent}$$