September 16 Math 2306 sec. 52 Fall 2022

Section 6: Linear Equations Theory and Terminology

We were considering an n^{th} order, linear, homogeneous ODE.

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

where all a_i are continuous and $a_n(x)\neq 0$ for all x in an interval I.

The **Principle of Superposition** states that if we have some solutions $y_1, y_2, \dots, y_k$ to this linear, homogeneous ODE, then every linear combination

$$y=c_1y_1+c_2y_2+\cdots+c_ky_k$$

is also a solution. We'll use this to build what we'll call the **general solution.**

Linear Dependence or Independence

Suppose we have a set of functions $f_1, f_2, \dots, f_n$ all defined on the same interval I, and consider the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \text{for all } x \text{ in } I.$$

- If there exists a set of constants, $c_1, \dots, c_n$ **not all zero** that make this equation true, then the functions $f_1, \dots, f_n$ are said to be **linearly dependent**.
- If this equation can **ONLY** be made true by taking $c_1 = c_2 = \cdots = c_n = 0$, then the functions $f_1, \dots, f_n$ are said to be **linearly independent**.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We showed that the equation

$$c_1 \sin x + c_2 \cos x = 0 \quad \text{for all real } x$$

is only true when $c_1 = 0$ and $c_2 = 0$.

Remark: If there are only two functions (like this case), the functions are linearly **dependent** if one is a constant multiple of the other.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

We determined that this set was linearly dependent because $f_3(x) = \frac{1}{4}f_2(x) - f_1(x)$. So we can write

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

which is true for all x. This is a linear combination with coefficients

$$c_1 = 1, \quad c_2 = -\frac{1}{4}, \quad \text{and} \quad c_3 = 1.$$

Since at least one of these numbers is not zero, this establishes linear dependence.

Linear Dependence Relation

An equation with at least one c nonzero, such as

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

from this last example is called a **linear dependence relation** for the functions $\{f_1, f_2, f_3\}$.

Example

Determine whether the set of functions $\{e^{2x}, e^{2x+1}\}$ is linearly dependent or linearly independent on $(0, \infty)$.

Let $f_1(x) = e^{2x}$ and $f_2(x) = e^{2x+1}$

Note that $f_2(x) = e^{2x} \cdot e^{1} = e\, e^{2x}$

so $f_2(x) = e\, f_1(x)$ a constant times f_1

$$-e f_1(x) + f_2(x) = 0$$

is a linear dependence relation.

$\{e^{2x}, e^{2x+1}\}$ is a linearly dependent set.

Definition of Wronskian

Definition: Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

The matrix will be 2x2 since there are two functions.

$$f_1'(x) = \cos x$$

$$f_2'(x) = -\sin x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$= -1$$

for $f_1(x) = \sin x$ and $f_2(x) = \cos x$

$$W(f_1, f_2)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

The matrix will be 3x3.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8-0) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(0-8)$$

$$= -8x^2 - 4x(-4x-2+4x) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

for $f_1(x) = x^2$, $f_2(x) = 4x$, $f_3(x) = x - x^2$

Theorem (a test for linear independence)

Theorem: Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

"If $W=0$ then linearly dependent"

Alternative Version

If y_1, $y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^{x}\left(-2e^{-2x}\right) - e^{x}\left(e^{-2x}\right)$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x}$$

which is not zero.

Hence y_1 and y_2 are linearly independent.

Fundamental Solution Set

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.