September 17 Math 2306 sec. 52 Fall 2021

Section 7: Reduction of Order

We were considering a **second order, linear, homogeneous** ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

for which one solution $y_1(x)$ is known.

Reduction of Order Formula

Reduction of order was a way of finding a second, linearly independent solution $y_2(x)$. We assumed that

$$y_2(x) = u(x)y_1(x).$$

We obtained the formula for u

$$u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx.$$

So our second solution

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx,$$

and the general solution

$$y = c_1 y_1 + c_2 y_2.$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Let's assume $x > 0$. $y_2 = uy_1$ where

$$u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\, dx.$$ Is $P(x) = -3x$?

no, ODE isn't in standard form

Standard form $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$

$P(x) = \frac{-3}{x}$, $-\int P(x)\,dx = -\int \frac{-3}{x}\,dx = 3\int \frac{1}{x}\,dx = 3\ln x$

$= \ln x^3$

$$\Rightarrow \quad e^{-\int P(x)\,dx} = e^{\ln x^3} = x^3$$

$$(y_1)^2 = (x^2)^2 = x^4$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2 = u y_1 = (\ln x)x^2 = x^2 \ln x$$

The general solution

$$\boxed{y = c_1 x^2 + c_2 x^2 \ln x}$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \quad \text{with } a \neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant})\, y' + (\text{constant})\, y?$$

y could be exponential, "e to a constant times x"

another possibility is Sine or Cosine.

We look for solutions of the form $y = e^{mx}$ with m constant.

$y = e^{mx}$ is supposed to solve $ay'' + by' + cy = 0$.

Let's sub it in.

$$y = e^{mx} \qquad ay'' + by' + cy = 0$$

$$y' = me^{mx} \qquad a(m^2e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$y'' = m^2e^{mx}.$$

$$e^{mx}(am^2 + bm + c) = 0$$

This is true if $am^2 + bm + c = 0$

This is called the characteristic (a.k.a. auxiliary) equation.

am^2+bm+c is the characteristic polynomial for the ODE

$$ay''+by'+cy=0.$$

So $y=e^{mx}$ is a solution if m is a root of the characteristic polynomial.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

2nd order, linear, homogeneous constant coef.

The characteristic equation is

$$m^2 - 2m - 2 = 0$$

Find the roots $\quad m^2 - 2m + 1 - 1 - 2 = 0$

$$(m-1)^2 - 3 = 0$$

$$(m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3}$$

$$m = 1 \pm \sqrt{3}$$

We have two distinct real roots

$$m_1 = 1 + \sqrt{3} \quad \text{and} \quad m_2 = 1 - \sqrt{3}.$$

Hence $y_1 = e^{(1+\sqrt{3})x}$ and $y_2 = e^{(1-\sqrt{3})x}$

The general solution

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$P(x) = \dfrac{b}{a}$ $\quad -\int P(x)\,dx = -\int \frac{b}{a}\,dx = \dfrac{-b}{a}x$

$$e^{-\int P(x)dx} = e^{-\frac{b}{a}x}, \quad (y_1)^2 = \left(e^{-\frac{b}{2a}x}\right)^2 = e^{-\frac{2b}{2a}x} = e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int 1\,dx = x$$

$$y_2 = uy_1 = x\,e^{-\frac{b}{2a}x}$$

The general solution

$$y = c_1 e^{-\frac{b}{2a}x} + c_2 x\, e^{-\frac{b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Example

Solve the ODE $4y'' - 4y' + y = 0$.

2^{nd} order, linear, homogeneous constant coef.

The characteristic equation is

$$4m^2 - 4m + 1 = 0$$

factors $(2m-1)^2 = 0$

$2m - 1 = 0 \Rightarrow m = \frac{1}{2}$ repeated

$y_1 = e^{\frac{1}{2}x}$ and $y_2 = x e^{\frac{1}{2}x}$

$\boxed{\text{The general solution } y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}.}$