September 17 Math 3260 sec. 51 Fall 2025

2.3 Matrices

Row Echelon form (ref):

1. Any row of all zero is below all rows containing a leading entry, and
2. each leading entry is to the right of the leading entry is every row above it.

Reduced Row Echelon form (rref): An ref such that

3. each leading entry is 1 (called a leading one), and
4. each leading one is the only nonzero entry is its column.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not an echelon form — an ref — an rref

Elementary Row Operations

- Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.
- Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.
- Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalence

Definition: We will say that two matrices are **row equivalent** if one matrix can be obtained from the other by performing some sequence of elementary row operations.

Theorem:

If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Uniqueness of an rref

Theorem:

A matrix A is row equivalent to exactly one reduced echelon form.

Remark: A matrix can be row equivalent to lots of different refs, but to only one RREF. So it makes sense to call it THE rref, and to write

$$\text{rref}(A).$$

Pivot Position & Pivot Column

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identifying Pivot Positions and Columns

The following matrices are **row equivalent**. Identify the pivot positions and pivot columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

B is an rref, B = rref(A). We can circle the pivot positions in A.

Pivot columns are 1, 2, and 4.

Complete Row Reduction isn't needed to find Pivots

The following three matrices are row equivalent. (Note, B is an ref but not an rref, and C is an rref.)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivot positions are the same is all three matrices. They're

- Not obvious from looking at A,
- but are obvious from looking at either B or C.

2.4 Solutions of Linear Systems

Recall: Row equivalent matrices correspond to equivalent systems.

Let's use an augmented matrix and its rref to characterize the solution set of the system

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$x_1 \quad -2x_3 + 3x_4 = -24$$

$$x_2 - 2x_3 + 2x_4 = -7$$

$$x_5 = 4$$

Solve for x_1, x_2 and x_5.

$$x_1 = -24 + 2x_3 - 3x_4$$

$$x_2 = -7 + 2x_3 - 2x_4$$

$$x_5 = 4$$

x_3, x_4 are any real number.

We note that the variables corresponding to pivot columns appear in exactly one equation. They also have a coefficient of 1 making them very easy to isolate. We will classify variables according to pivot and non-pivot columns.

Basic & Free Variables

Definition: Let A be an $m \times n$ matrix that is the coefficient matrix for a system of linear equations in the n variables, $x_1, x_2, \ldots, x_n$. For each $i = 1, \ldots, n$

- if the i^{th} column of A is a pivot column, then x_i is a **basic variable**, and
- if the i^{th} column of A is not a pivot column, the x_i is a **free variable**.

Basic & Free Variables

Consider the system of equations along with its augmented matrix.

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

This is row equivalent to the following rref.

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Hence the **basic** variables are x_1, x_2, and x_5, and the **free** variables are x_3 and x_4.

Expressing Solutions

To avoid confusion, i.e., in the interest of clarity, we will **always** write solution sets by expressing basic variables in terms of free variables. We will not write free variables in terms of basic. That is, the solution set to the system whose augmented matrix is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 \quad -2x_3 \quad = 3$

$x_1 = 3 + 2x_3$

will be written

$$\begin{array}{lcl} x_1 & = & 3+2x_3 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array}, \quad \text{i.e.,} \quad \begin{array}{lcl} x_1 & = & 3+2t \\ x_2 & = & 2+2t \\ x_3 & = & t \\ x_4 & = & 0 \end{array}, \quad t \in R.$$

Proper Solution Set Expressions

We will never express free variables in terms of basic variables. All three of the following result from the same augmented matrix:

$$\begin{array}{rcl} x_1 & = & 3+2t \\ x_2 & = & 2+2t \\ x_3 & = & t \\ x_4 & = & 0 \end{array} \qquad \begin{array}{rcl} x_1 & = & 3+2x_3 \\ x_2 & = & 2+2x_3 \\ x_3 & = & -\frac{3}{2}+\frac{1}{2}x_1 \\ x_4 & = & 0 \end{array} \qquad \begin{array}{rcl} x_1 & = & 3+2x_3 \\ x_2 & = & -1+x_1 \\ x_3 & = & -1+\frac{1}{2}x_2 \\ x_4 & = & 0 \end{array}$$

The left most parametric description is correct. The two expressions in red are **not correct** descriptions. They both include convoluted descriptions of the relationships between the variables.

Main Existence & Uniqueness Theorem

Let A and $\widehat{A}$ be the coefficient matrix and the augmented matrix, respectively of a system of linear equations.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then the system is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then the system is consistent.

Moreover, if the system is consistent, then

1. if every column of A is a pivot column of A, then the system has a unique solution; and
2. if at least one column of A is not a pivot column of A, then the system has infinitely many solutions.

$A =$ coefficient matrix and $\widehat{A} =$ augmented matrix

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then the system is inconsistent.

If the right most column of $\widehat{A}$ is a pivot column, then the augmented column contains a leading 1. This means that the rref has a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

The system is equivalent to one with an equation

$$0 = 1.$$

2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then the system is consistent.

If the right most column of $\widehat{A}$ is NOT pivot column, then there are no false equations.

A = coefficient matrix and $\widehat{A}$ = augmented matrix

Suppose the system is consistent.

1. If every column of A is a pivot column of A, then the system has a unique solution.

If each column of A is a pivot column, then every variable is a basic variable.

2. If at least one column of A is not a pivot column of A, then the system has infinitely many solutions.

Each non-pivot column corresponds to a free variable. Free variables give rise to infinitely many solutions.

Examples of each case:

$$\text{rref}(\widehat{A}) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{rref}(A) = \left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

inconsistent

$$\text{rref}(\widehat{A}) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{rref}(A) = \left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

consistent
infinite solns.

$$\text{rref}(\widehat{A}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \text{rref}(A) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

consistent
one soln.

Remark: If the system is **homogeneous**, the augmented column is all zero. The right most column **cannot be a pivot column**. Again, homogeneous systems are **always consistent**!

Caveat

It's tempting to think row of zeros = infinitely many solutions. But that's **not true!** What is true is

Consistent with free variables = infinitely many solutions.

A free variables is indicated by a **non-pivot column** not by a zero row.

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{inconsistent}}, \quad \underbrace{\left[\begin{array}{ccr|c} 1 & 0 & 4 & 2\\ 0 & 1 & -7 & 5\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{consistent } \infty\text{-many solns}}, \quad \underbrace{\left[\begin{array}{ccc|r} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{consistent one soln}}$$