

September 18 Math 2306 sec. 51 Fall 2024

Section 6: Linear Equations Theory and Terminology

We were focusing on the homogeneous, linear ODE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0, \quad (1)$$

and assuming that all $a_i(x)$ are continuous on some interval I and that $a_n(x) \neq 0$ for all x in I .

We stated the **Principle of Superposition** that says that if we have solutions y_1, \dots, y_k of (1), then every linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k$$

is also a solution.

Linear Dependence/Independence

Definition:

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants c_1, c_2, \dots, c_n with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \quad \text{for all } x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I .

We found, for example, that the set $\{\sin x, \cos x\}$ is **linearly independent**, and the set $\{x^2, 4x, x - x^2\}$ is **linearly dependent** on the interval $(-\infty, \infty)$.

Definition of Wronskian

We also defined the Wronskian last time:

Definition: Wronskian

Let f_1, f_2, \dots, f_n possess at least $n - 1$ continuous derivatives on an interval I . The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \dots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Theorem (a test for linear independence)

Let f_1, f_2, \dots, f_n be $n - 1$ times continuously differentiable on an interval I . If there exists x_0 in I such that

$$W(f_1, f_2, \dots, f_n)(x_0) \neq 0,$$

then the functions are **linearly independent** on I .

Remark: For the sorts of functions we're interested in¹, we can use this as a test:

$$W = 0 \implies \text{dependent} \quad \text{or} \quad W \neq 0 \implies \text{independent}$$

¹It is possible to create a set of linearly independent functions having a zero Wronskian. It is true that if $W \neq 0$, the functions are definitely independent.

Example

Determine whether the functions are linearly dependent or linearly independent on the give interval.

$$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$$

We can use the Wronskian.

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= x^2(3x^2) - 2x(x^3) = \\ &= 3x^4 - 2x^4 \\ &= x^4 \end{aligned}$$

$$W(y_1, y_2)(x) = x^4 \neq 0$$

Hence y_1 and y_2 are linearly independent.

Fundamental Solution Set

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

We're ready to get at what the solution to a homogeneous linear ODE will be. First, a definition.

Definition: Fundamental Solution Set

A set of functions y_1, y_2, \dots, y_n is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

- (i) are solutions of the equation,
- (ii) there are n of them, and
- (iii) they are linearly independent.

Fundamental Solution Set

Theorem

If a_1, a_2, \dots, a_n are continuous on an interval I and $a_n(x) \neq 0$ for every x in I , then the homogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

possess a fundamental solution set.

So under the conditions on the coefficients that we've stated, a fundamental solution exists. The next definition tells us what the general solution to the ODE is.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (2)$$

Definition: General Solution of Homogeneous, Linear ODE

Let y_1, y_2, \dots, y_n be a fundamental solution set of the n^{th} order linear homogeneous equation (2). Then the **general solution** of (2) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_1, c_2, \dots, c_n are arbitrary constants.

Remark: This indicates that the task of solving an n^{th} order linear **homogeneous** ODE is to find a fundamental solution set, i.e., n , linearly independent solutions. We build the general solution by creating a linear combination.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2 y'' - 4xy' + 6y = 0 \quad \text{on } (0, \infty),$$

and determine the general solution.

We need to show that the two functions y_1 and y_2 are linearly independent solutions.

Is y_1 a solution? sub it in.

$$y_1 = x^2$$

$$y_1' = 2x$$

$$y_1'' = 2$$

$$x^2 y_1'' - 4x y_1' + 6y_1 \stackrel{?}{=} 0$$

$$x^2(2) - 4x(2x) + 6(x^2) \stackrel{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

yes, y_1 is a solution

$$x^2 y'' - 4xy' + 6y = 0$$

Is y_2 a solution?

$$y_2 = x^3$$

$$y_2' = 3x^2$$

$$y_2'' = 6x$$

$$x^2 y_2'' - 4x y_2' + 6y_2 \stackrel{?}{=} 0$$

$$x^2(6x) - 4x(3x^2) + 6(x^3) \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$$

$$0 = 0$$

Both y_1 and y_2 are solutions.

We found that $W(y_1, y_2)(x) = x^4$
which is non zero. Hence they are
linearly independent.

So y_1, y_2 is a fundamental solution set.

The general solution is

$$y = C_1 x^2 + C_2 x^3$$