September 18 Math 2306 sec. 53 Fall 2024

Section 6: Linear Equations Theory and Terminology

We were focusing on the homogeneous, linear ODE

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0, \qquad (1)$$

and assuming that all $a_i(x)$ are continuous on some interval I and that $a_n(x)\neq 0$ for all x in I.

We stated the Principle of Superposition that says that if we have solutions $y_1,\ldots,y_k$ of (1), then every linear combination

$$y=c_1y_1+c_2y_2+\cdots+c_ky_k$$

is also a solution.

Linear Dependence/Independence

Definition:

A set of functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants c_1, $c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

We found, for example, that the set $\{\sin x, \cos x\}$ is **linearly independent**, and the set $\{x^2, 4x, x - x^2\}$ is **linearly dependent** on the interval $(-\infty, \infty)$.

Definition of Wronskian

We also defined the Wronskian last time:

Definition: Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Theorem (a test for linear independence)

Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that

$$W(f_1, f_2, \ldots, f_n)(x_0) \neq 0,$$

then the functions are **linearly independent** on I.

Remark: For the sorts of functions we're interested in[1], we can use this as a test:

$$W = 0 \Longrightarrow \text{dependent} \quad \text{or} \quad W \neq 0 \Longrightarrow \text{independent}$$

[1] It is possible to create a set of linearly independent functions having a zero Wronskian. It is true that if $W \neq 0$, the functions are definitely independent.

Example

Determine whether the functions are linearly dependent or linearly independent on the give interval.

$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

$$W(y_1, y_2)(x) = x^4, \quad y_1 = x^2, \quad y_2 = x^3$$

Since this is nonzero,

y_1 and y_2 are linearly independent.

Fundamental Solution Set

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

We're ready to get at what the solution to a homogeneous linear ODE will be. First, a definition.

Definition: Fundamental Solution Set

A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Fundamental Solution Set

Theorem

If $a_1, a_2, \dots, a_n$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I, then the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

possess a fundamental solution set.

So under the conditions on the coefficients that we've stated, a fundamental solution exists. The next definition tells us what the general solution to the ODE is.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{2}$$

Definition: General Solution of Homogeneous, Linear ODE

Let $y_1,\ y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation (2). Then the **general solution** of (2) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1,\ c_2, \ldots, c_n$ are arbitrary constants.

Remark: This indicates that the task of solving an n^{th} order linear **homogeneous** ODE is to find a fundamental solution set, i.e., n, linearly independent solutions. We build the general solution by creating a linear combination.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2 y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

we need to show that the two functions y_1 and y_2 are solutions and are linearly independent. Let's verify that they are solutions.

Is y_1 a solution?

$y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$x^2 y_1'' - 4x y_1' + 6y_1 \overset{?}{=} 0$

$x^2(2) - 4x(2x) + 6(x^2) \overset{?}{=} 0$

$2x^2 - 8x^2 + 6x^2 \overset{?}{=} 0$

y_1 is a solution. $0 = 0$ ✓

$$x^2y'' - 4xy' + 6y = 0$$

Is y_2 a solution?

$y_2 = x^3$.

$y_2' = 3x^2$

$y_2'' = 6x$

$x^2 y_2'' - 4x y_2' + 6 y_2 \stackrel{?}{=} 0$

$x^2(6x) - 4x(3x^2) + 6(x^3) \stackrel{?}{=} 0$

$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$

$0 = 0$

Both y_1 and y_2 are solutions.

We found that $W(y_1, y_2)(x) = x^4 \neq 0$

so they are linearly independent.

Hence y_1 and y_2 form a fundamental solution set.

The general solution is

$$y = c_1 x^2 + c_2 x^3$$