September 19 Math 2306 sec. 51 Fall 2022

Section 6: Linear Equations Theory and Terminology

We were considering an n^{th} order, linear, homogeneous ODE.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition: A **Fundamental Solution Set** for this homogeneous equation is a set of n linearly independent solutions.

Definition Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **General Solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We have to show that y_1 and y_2 are linearly independent solutions.

Let's verify that they are solutions.

y_1 : $\quad y_1 = x^2, \quad y_1' = 2x, \quad y_1'' = 2$

$$x^2 y_1'' - 4x y_1' + 6 y_1 \stackrel{?}{=} 0$$

$$x^2(2) - 4x(2x) + 6x^2 \stackrel{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

y_1 is a solution

y_2: $\quad y_2 = x^3, \quad y_2' = 3x^2, \quad y_2'' = 6x$

$$x^2 y_2'' - 4x y_2' + 6y_2 \stackrel{?}{=} 0$$

$$x^2(6x) - 4x(3x^2) + 6x^3 \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

y_2 is also a solution

Let's show that y_1 and y_2 are linearly independent.

Let's use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4$$

Since this is not zero, y_1 and y_2 are linearly independent. We have 2 solutions that are lin. independent, hence we have a fundamental solution set.

$$\boxed{\text{The general solution is} \quad y = c_1 x^2 + c_2 x^3}$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Theorem: Let y_p be any solution of the nonhomogeneous equation, and let $y_1, y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.
Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_1(x)+g_2(x) \quad (1)$$

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_0(x)y=g_2(x),$$

then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution for the nonhomogeneous equation (1).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$$y_{p_1} = 6, \quad y_{p_1}' = 0, \quad y_{p_1}'' = 0$$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x, \quad y_{p_2}' = -7, \quad y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x\, y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y = y_c + y_p$ $\qquad$ $y_c = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^3$

And from (a) and (b) and the principle of superposition, $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

$$\boxed{\text{The general solution is } y = c_1 x^2 + c_2 x^3 + 6 - 7x}$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution $y = c_1x^2 + c_2x^3 + 6 - 7x$.

we need to find c_1 and c_2 to satisfy the IC.

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$

$$y' = 2c_1x + 3c_2x^2 - 7$$

$$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = -5$$

$$2C_1 + 3C_2 = 2$$

Solve $C_1 + C_2 = 1$ $\Rightarrow$ $2C_1 + 2C_2 = 2$

$2C_1 + 3C_2 = 2$ $\quad$ $-$ $2C_1 + 3C_2 = 2$

$$-C_2 = 0 \qquad C_2 = 0$$

$$C_1 + 0 = 1 \Rightarrow C_1 = 1$$

The solution to the IVP is

$$y = x^2 + 6 - 7x$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order.** Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x) y_1(x).$$

By linear independence u can't be constant

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form
$$y_2(x)=u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u <u>cannot</u> be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

Put the ODE in standard form:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$$

Set $y_2 = uy_1$

$y_2 = xu$ substitute

$y_2' = xu' + u$

$y_2'' = xu'' + u' + u' = xu'' + 2u'$

$$y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 = 0$$

$$x u'' + 2u' - \frac{1}{x}(x u' + u) + \frac{1}{x^2}(x u) = 0$$

$$x u'' + 2u' - u' - \cancel{\frac{1}{x} u} + \cancel{\frac{1}{x} u} = 0$$

$$x u'' + u' = 0$$

Let $w = u'$ so that $w' = u''$

$$x w' + w = 0$$ 1st order linear and separable

Separating:

$$x \frac{dw}{dx} = -w$$

$$\frac{1}{w} \frac{dw}{dx} = \frac{-1}{x}$$

$$\int \frac{1}{w}\,dw = \int \frac{-1}{x}\,dx$$

$$\ln|w| = -\ln x + C = \ln x^{-1} + C$$

$$|w| = e^{\ln x^{-1} + C} = e^{C} x^{-1}$$

Choose $k = \pm e^{C}$

$$w = k x^{-1}$$

$$u' = w \Rightarrow \quad u = \int w\,dx = k \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad u = k \ln x$$

So $y_2 = xu = kx\ln x$

we can let $k=1$

$$y_2 = x\ln x$$

The general solution

$$y = c_1 x + c_2 x\ln x$$