

2.4 Solutions of Linear Systems

Recall: We defined a **pivot position** in a matrix A as the location (row/column) of a leading one in $\text{rref}(A)$. A pivot column is a column containing a pivot position.

Also Recall: If A is the coefficient for a linear system of equations in the variables x_1, \dots, x_n , then

- ▶ x_i is a **basic variable** if the i^{th} column of A is a pivot column, and
- ▶ x_i is a **free variable** if the i^{th} column of A is not a pivot column.

Main Existence & Uniqueness Theorem

Let A and \hat{A} be the coefficient matrix and the augmented matrix, respectively of a system of linear equations.

1. If the rightmost column of \hat{A} is a pivot column of \hat{A} , then the system is inconsistent.
2. If the rightmost column of \hat{A} is not a pivot column of \hat{A} , then the system is consistent.

Moreover, if the system is consistent, then

1. if every column of A is a pivot column of A , then the system has a unique solution; and
2. if at least one column of A is not a pivot column of A , then the system has infinitely many solutions.

Example

Find an equation in g , h , and k such that the given matrix is the augmented matrix of a consistent linear system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ -3 & -2 & 0 & h \\ 2 & 1 & -2 & k \end{array} \right]$$

We need to determine whether the right most column is a pivot column.

Do row ops. $3R_1 + R_2 \rightarrow R_2$ and $-2R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ 0 & 4 & 24 & h+3g \\ 0 & -3 & -18 & k-2g \end{array} \right]$$

$$\begin{array}{cccc} 3 & 6 & 24 & 3g \\ -3 & -2 & 0 & h \\ -2 & -4 & -16 & -2g \\ 2 & 1 & -2 & k \end{array}$$

$$\frac{1}{4} R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 9 \\ 0 & 1 & 6 & \frac{1}{4}(h+3g) \\ 0 & -3 & -18 & k-2g \end{array} \right]$$

$$3 R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 9 \\ 0 & 1 & 6 & \frac{1}{4}(h+3g) \\ 0 & 0 & 0 & k-2g+\frac{3}{4}(h+3g) \end{array} \right]$$

The system will only be consistent if

$$k-2g+\frac{3}{4}(h+3g)=0$$

Let's simplify this.

$$4k - 8g + 3(h + 3g) = 0$$

$$4k - 8g + 3h + 9g = 0$$

$$\Rightarrow 4k + 3h + g = 0$$

This is an equation in g , h , and k . The matrix is the augmented matrix of a **consistent** linear system only if this is true.

Chapter 3: Matrix Algebra

In this chapter, we're going to consider **matrices** as algebraic objects in their own right. We'll eventually want to define and work with several operations:

- ▶ scalar multiplication,
- ▶ matrix addition,
- ▶ matrix multiplication (i.e., multiplying two matrices),
- ▶ transposition, and
- ▶ inversion.

We will also work with matrices and vectors together by defining multiplication of a vector by a matrix.

3.1 Notation, Row & Column Vectors

Consider a generic $m \times n$ matrix A .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We can use the short hand

$$A = [a_{ij}]$$

to refer to this matrix. The element in row i and column j is called an **entry**. We can refer to it using either of the following notations:

$$a_{ij} \quad \text{or} \quad A_{(i,j)}.$$

Caveat: The notation A_{ij} should not be used in place of $A_{(i,j)}$ because it is used to mean something else!

Example

$$A = \begin{bmatrix} -3 & -2 & -1 & 2 \\ -3 & -4 & 5 & -3 \\ 4 & 7 & 3 & -5 \end{bmatrix}$$

Identify each of the following, if it exists:

1. $a_{32} = 7$

2. $a_{23} = 5$

3. $A_{(1,4)} = 2$

4. $A_{(4,1)}$ there is not one

Row & Column Vectors

For $m \times n$ matrix $A = [a_{ij}]$, we define the **row vectors** of A to be the m vectors in R^n given by

$$\begin{aligned}\text{Row}_1(A) &= \langle a_{11}, a_{12}, \dots, a_{1n} \rangle \\ \text{Row}_2(A) &= \langle a_{21}, a_{22}, \dots, a_{2n} \rangle \\ &\vdots \\ \text{Row}_m(A) &= \langle a_{m1}, a_{m2}, \dots, a_{mn} \rangle\end{aligned}$$

Similarly, we define the **column vectors** of A to be the n vectors in R^m given by

$$\begin{aligned}\text{Col}_1(A) &= \langle a_{11}, a_{21}, \dots, a_{m1} \rangle \\ \text{Col}_2(A) &= \langle a_{12}, a_{22}, \dots, a_{m2} \rangle \\ &\vdots \\ \text{Col}_n(A) &= \langle a_{1n}, a_{2n}, \dots, a_{mn} \rangle\end{aligned}$$

Example

Let $A = \begin{bmatrix} -1 & -9 & 8 & 4 \\ -5 & 10 & -2 & 5 \\ 7 & 7 & -7 & 4 \end{bmatrix}$. Identify the row vectors of A .

$$\text{Row}_1(A) = \langle -1, -9, 8, 4 \rangle$$

$$\text{Row}_2(A) = \langle -5, 10, -2, 5 \rangle$$

$$\text{Row}_3(A) = \langle 7, 7, -7, 4 \rangle.$$

Example

Let $A = \begin{bmatrix} -1 & -9 & 8 & 4 \\ -5 & 10 & -2 & 5 \\ 7 & 7 & -7 & 4 \end{bmatrix}$. Identify the column vectors of A .

$$\text{Col}_1(A) = \langle -1, -5, 7 \rangle$$

$$\text{Col}_2(A) = \langle -9, 10, 7 \rangle$$

$$\text{Col}_3(A) = \langle 8, -2, -7 \rangle$$

$$\text{Col}_4(A) = \langle 4, 5, 4 \rangle$$

3.2 Addition, Subtraction, Scalar Multiplication

Definition of Matrix Addition

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices. Then their sum, written $A + B$, is defined to be the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

Definition of Scalar Multiplication

Suppose $A = [a_{ij}]$ is an $m \times n$ matrices, and let c be a scalar. Then we define the scalar multiple, written cA , as the $m \times n$ matrix

$$cA = [ca_{ij}].$$

Remark 1: The sum $A + B$ is only defined if A and B are the same size.

Remark 2: We can define the difference $A - B = A + (-1)B$.

Example:

$$\text{Let } A = \begin{bmatrix} -1 & 4 & -4 & 1 \\ -3 & 1 & 1 & -2 \\ 0 & 3 & 0 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 4 & -3 \\ -4 & 2 & -4 & 1 \\ -4 & -1 & -1 & 3 \end{bmatrix}.$$

Evaluate

$$1. A + B = \begin{bmatrix} -1+2 & 4+4 & -4+4 & 1-3 \\ -3-4 & 1+2 & 1-4 & -2+1 \\ 0-4 & 3-1 & 0-1 & -4+3 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 0 & -2 \\ -7 & 3 & -3 & -1 \\ -4 & 2 & -1 & -1 \end{bmatrix}$$

$$2. 4A = \begin{bmatrix} -4 & 16 & -16 & 4 \\ -12 & 4 & 4 & -8 \\ 0 & 12 & 0 & -16 \end{bmatrix}$$

$$3. A - B = \begin{bmatrix} -3 & 0 & -8 & 4 \\ 1 & -1 & 5 & -3 \\ 4 & 4 & 1 & -7 \end{bmatrix}$$

The $m \times n$ matrix all of whose entries are zero will be denoted $O_{m \times n}$. We'll call this the $m \times n$ **zero matrix**.

Algebraic Properties

Let A , B , and C be $m \times n$ matrices, and let c and d be scalars. Then

$$(i) \ A + B = B + A$$

$$(v) \ c(A + B) = cA + cB$$

$$(ii) \ (A + B) + C = A + (B + C)$$

$$(vi) \ (c + d)A = cA + dA$$

$$(iii) \ A + O_{m \times n} = A$$

$$(vii) \ c(dA) = d(cA) = (cd)A$$

$$(iv)^a \ A + (-A) = O_{m \times n}$$

$$(viii) \ 1A = A$$

^aThe term $-A$ denotes $(-1)A$.

Matrix Addition & Scalar Multiplication

On Row & Column Vectors

We can note how matrix addition and scalar multiplication of matrices affects row and column vectors. In particular, if A and B are $m \times n$ matrices and c is a scalar, then

- ▶ $\text{Row}_i(A + B) = \text{Row}_i(A) + \text{Row}_i(B)$ for each $i = 1, \dots, m$
- ▶ $\text{Col}_j(A + B) = \text{Col}_j(A) + \text{Col}_j(B)$ for each $j = 1, \dots, n$
- ▶ $\text{Row}_i(cA) = c \text{Row}_i(A)$ for each $i = 1, \dots, m$, and
- ▶ $\text{Col}_j(cA) = c \text{Col}_j(A)$ for each $j = 1, \dots, n$

3.3 Multiplication of Two Matrices

The Product of Matrices

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix

$$AB = [(AB)_{(i,j)}], \quad \text{where} \quad (AB)_{(i,j)} = \text{Row}_i(A) \cdot \text{Col}_j(B).$$

If the number of columns of A does not match the number of rows of B , then AB is not defined.

Remark: This says that the entry in row i and column j of the product AB is the dot product of the i^{th} row vector of A and the j^{th} column vector of B . These vectors must be the same size for the dot product to make sense.

Schematic for AB to be Defined

If we write the matrices in the order of the product, and write their sizes underneath, then the middle numbers have to match.

$$\begin{array}{c} A \quad B \\ m \times p \quad p \times n \\ \swarrow \quad \searrow \\ m \times n \end{array}$$

The diagram illustrates the compatibility condition for matrix multiplication. Matrix A has dimensions $m \times p$ and matrix B has dimensions $p \times n$. The middle p in both dimensions is highlighted in green. A bracket connects these two p 's, with an equals sign $(=)$ below it, indicating they must be equal for the product to be defined. Arrows point from the m and n dimensions to the final product dimensions $m \times n$.

The size of the product will be the outer numbers.

Example

Suppose A is 3×5 and B is 5×2 .

1. Is AB defined? If so, what is its size?

$$\begin{array}{cc} A & B \\ 3 \times 5 & 5 \times 2 \\ \downarrow & \downarrow \\ & 3 \times 2 \end{array}$$

AB is defined.

AB is 3×2 .

2. Is BA defined? If so, what is its size?

$$\begin{array}{cc} B & A \\ 5 \times 2 & 3 \times 5 \\ \neq & \end{array}$$

BA is not defined.

It should be clear from these examples that order is important in matrix multiplication. That is, it doesn't commute.

Example

Suppose C is 6×4 and D is 4×6 .

1. Is CD defined? If so, what is its size?

$$\begin{array}{ccc} CD & & \\ 6 \times 4 & 4 \times 6 & \\ = & & \\ \downarrow & \downarrow & \\ 6 \times 6 & & \end{array}$$

CD is defined
it is 6×6 .

2. Is DC defined? If so, what is its size?

$$\begin{array}{ccc} DC & & \\ 4 \times 6 & 6 \times 4 & \\ = & & \\ \downarrow & \downarrow & \\ 4 \times 4 & & \end{array}$$

DC is defined
it is 4×4 .