September 19 Math 3260 sec. 53 Fall 2025

2.4 Solutions of Linear Systems

Recall: We defined a **pivot position** in a matrix A as the location (row/column) of a leading one in $\mathrm{rref}(A)$. A pivot column is a column containing a pivot position.

Also Recall: If A is the coefficient for a linear system of equations in the variables $x_1, \ldots, x_n$, then

- x_i is a **basic variable** if the i^{th} column of A is a pivot column, and
- x_i is a **free variable** if the i^{th} column of A is a not a pivot column.

Main Existence & Uniqueness Theorem

Let A and $\widehat{A}$ be the coefficient matrix and the augmented matrix, respectively of a system of linear equations.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then the system is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then the system is consistent.

Moreover, if the system is consistent, then

1. if every column of A is a pivot column of A, then the system has a unique solution; and
2. if at least one column of A is not a pivot column of A, then the system has infinitely many solutions.

Example

Find an equation in g, h, and k such that the given matrix is the augmented matrix of a consistent linear system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ -3 & -2 & 0 & h \\ 2 & 1 & -2 & k \end{array}\right]$$

We need to determine what condition(s) there are on g, h, k so that the last column is not a pivot column.

We can do row ops.

$3R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ -3 & -2 & 0 & h \\ 2 & 1 & -2 & k \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ 0 & 4 & 24 & h+3g \\ 0 & -3 & -18 & k-2g \end{array}\right]$$

$$\begin{array}{cccc} 3 & 6 & 24 & 3g \\ -3 & -2 & 0 & h \end{array}$$

$$\begin{array}{cccc} -2 & -4 & -16 & -2g \\ 2 & 1 & -2 & k \end{array}$$

$$\frac{1}{4}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ 0 & 1 & 6 & \frac{1}{4}(h+3g) \\ 0 & -3 & -18 & k-2g \end{array}\right]$$

$$3R_2 + R_3 \to R_3$$

$$\begin{array}{cccc} 0 & 3 & 18 & \frac{3}{4}(h+3g) \\ 0 & -3 & -18 & k-2g \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & g \\ 0 & 1 & 6 & \frac{1}{4}(h+3g) \\ 0 & 0 & 0 & k-2g+\frac{3}{4}(h+3g) \end{array}\right]$$

The last column is not a pivot column if

$$k - 2g + \frac{3}{4}(h + 3g) = 0$$

Simplifying,

$$4k - 8g + 3(h + 3g) = 0$$

$$4k - 8g + 3h + 9g = 0$$

$$4k + 3h + g = 0$$

This is an equation in g, h, and k. If this equation is true, then the system having that augmented matrix would be consistent.

Chapter 3: Matrix Algebra

In this chapter, we're going to consider **matrices** as algebraic objects in their own right. We'll eventually want to define and work with several operations:

- scalar multiplication,
- matrix addition,
- matrix multiplication (i.e., multiplying two matrices),
- transposition, and
- inversion.

We will also work with matrices and vectors together by defining multiplication of a vector by a matrix.

3.1 Notation, Row & Column Vectors

Consider a generic $m \times n$ matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We can use the short hand

$$A = [a_{ij}]$$

to refer to this matrix. The element in row i and column j is called an **entry**. We can refer to it using either of the following notations:

$$a_{ij} \quad \text{or} \quad A_{(i,j)}.$$

Caveat: The notation A_{ij} should not be used in place of $A_{(i,j)}$ because it is used to mean something else!

Example

$$A = \begin{bmatrix} -3 & -2 & -1 & 2 \\ -3 & -4 & 5 & -3 \\ 4 & 7 & 3 & -5 \end{bmatrix}$$

Identify each of the following, if it exists:

1. a_{32} = 7
2. a_{23} = 5
3. $A_{(1,4)}$ = 2
4. $A_{(4,1)}$ there is no $A_{(4,1)}$ a_{41}

Row & Column Vectors

For $m \times n$ matrix $A = [a_{ij}]$, we define the **row vectors** of A to be the m vectors in R^n given by

$$\begin{array}{rcl} \mathrm{Row}_1(A) & = & \langle a_{11}, a_{12}, \ldots, a_{1n} \rangle \\ \mathrm{Row}_2(A) & = & \langle a_{21}, a_{22}, \ldots, a_{2n} \rangle \\ \vdots & \vdots & \vdots \\ \mathrm{Row}_m(A) & = & \langle a_{m1}, a_{m2}, \ldots, a_{mn} \rangle \end{array}$$

Similarly, we define the **column vectors** of A to be the n vectors in R^m given by

$$\begin{array}{rcl} \mathrm{Col}_1(A) & = & \langle a_{11}, a_{21}, \ldots, a_{m1} \rangle \\ \mathrm{Col}_2(A) & = & \langle a_{12}, a_{22}, \ldots, a_{m2} \rangle \\ \vdots & \vdots & \vdots \\ \mathrm{Col}_n(A) & = & \langle a_{1n}, a_{2n}, \ldots, a_{mn} \rangle \end{array}$$

Example

Let $A = \begin{bmatrix} -1 & -9 & 8 & 4 \\ -5 & 10 & -2 & 5 \\ 7 & 7 & -7 & 4 \end{bmatrix}$. Identify the row vectors of A.

$\text{Row}_1(A) = \langle -1, -9, 8, 4 \rangle$

$\text{Row}_2(A) = \langle -5, 10, -2, 5 \rangle$

$\text{Row}_3(A) = \langle 7, 7, -7, 4 \rangle$

Example

Let $A = \begin{bmatrix} -1 & -9 & 8 & 4 \\ -5 & 10 & -2 & 5 \\ 7 & 7 & -7 & 4 \end{bmatrix}$. Identify the column vectors of A.

$Col_1(A) = \langle -1, -5, 7 \rangle$

$Col_2(A) = \langle -9, 10, 7 \rangle$

$Col_3(A) = \langle 8, -2, -7 \rangle$

$Col_4(A) = \langle 4, 5, 4 \rangle.$

3.2 Addition, Subtraction, Scalar Multiplication

Definition of Matrix Addition

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices. Then their sum, written $A + B$, is defined to be the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

Definition of Scalar Multiplication

Suppose $A = [a_{ij}]$ is an $m \times n$ matrices, and let c be a scalar. Then we define the scalar multiple, written cA, as the $m \times n$ matrix

$$cA = [ca_{ij}].$$

Remark 1: The sum $A + B$ is only defined if A and B are the same size.
Remark 2: We can define the difference $A - B = A + (-1)B$.

Example:

Let $A = \begin{bmatrix} -1 & 4 & -4 & 1 \\ -3 & 1 & 1 & -2 \\ 0 & 3 & 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 4 & -3 \\ -4 & 2 & -4 & 1 \\ -4 & -1 & -1 & 3 \end{bmatrix}$.

Evaluate

1. $A + B$ $= \begin{bmatrix} -1+2 & 4+4 & -4+4 & 1-3 \\ -3-4 & 1+2 & 1-4 & -2+1 \\ 0-4 & 3-1 & 0-1 & -4+3 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 0 & -2 \\ -7 & 3 & -3 & -1 \\ -4 & 2 & -1 & -1 \end{bmatrix}$

2. $4A$ $= \begin{bmatrix} -4 & 16 & -16 & 4 \\ -12 & 4 & 4 & -8 \\ 0 & 12 & 0 & -16 \end{bmatrix}$

3. $A - B$ $= \begin{bmatrix} -3 & 0 & -8 & 4 \\ 1 & -1 & 5 & -3 \\ 4 & 4 & 1 & -7 \end{bmatrix}$

The $m \times n$ matrix all of whose entries are zero will be denoted $O_{m\times n}$. We'll call this the $m \times n$ **zero matrix.**

Algebraic Properties

Let A, B, and C be $m \times n$ matrices, and let c and d be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O_{m\times n} = A$

(iv)[a] $A + (-A) = O_{m\times n}$

(v) $c(A + B) = cA + cB$

(vi) $(c + d)A = cA + dA$

(vii) $c(dA) = d(cA) = (cd)A$

(viii) $1A = A$

[a] The term $-A$ denotes $(-1)A$.

Matrix Addition & Scalar Multiplication

On Row & Column Vectors

We can note how matrix addition and scalar multiplication of matrices affects row and column vectors. In particular, if A and B are $m \times n$ matrices and c is a scalar, then

- $\mathrm{Row}_i(A+B) = \mathrm{Row}_i(A) + \mathrm{Row}_i(B)$ for each $i = 1, \ldots, m$
- $\mathrm{Col}_j(A+B) = \mathrm{Col}_j(A) + \mathrm{Col}_j(B)$ for each $j = 1, \ldots, n$
- $\mathrm{Row}_i(cA) = c\,\mathrm{Row}_i(A)$ for each $i = 1, \ldots, m$, and
- $\mathrm{Col}_j(cA) = c\,\mathrm{Col}_j(A)$ for each $j = 1, \ldots, n$

3.3 Multiplication of Two Matrices

The Product of Matrices

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix

$$AB = \left[(AB)_{(i,j)}\right], \quad \text{where} \quad (AB)_{(i,j)} = \text{Row}_i(A) \cdot \text{Col}_j(B).$$

If the number of columns of A does not match the number of rows of B, then AB is not defined.

Remark: This says that the entry in row i and column j of the product AB is the dot product of the i^{th} row vector of A and the j^{th} column vector of B. These vectors must be the same size for the dot product to make sense.

Schematic for *AB* to be Defined

If we write the matrices in the order of the product, and write their sizes underneath, then the middle numbers have to match.

$$A \quad B$$

$$m \times \underbrace{p \; p}_{(=)} \times n$$

$$\searrow \qquad \swarrow$$

$$m \times n$$

The size of the product will be the outer numbers.

Example

Suppose A is 3×5 and B is 5×2.

1. Is AB defined? If so, what is its size?

$A \quad B$
$3 \times 5 \;\checkmark\; 5 \times 2$
$\searrow \quad \swarrow$
3×2

AB is defined.

AB is 3×2

2. Is BA defined? If so, what is its size?

$B \quad A$
$5 \times 2 \quad 3 \times 5$
$\neq$

BA is not defined

Example

Suppose C is 6×4 and D is 4×6.

1. Is CD defined? If so, what is its size?

CD
$6\times 4 \quad 4\times 6$
6×6

CD is defined.
It's 6×6

2. Is DC defined? If so, what is its size?

DC
$4\times 6 \quad 6\times 4$
4×4

DC is defined
It is 4×4.