September 1 Math 2306 sec. 52 Fall 2021

Section 4: First Order Equations: Linear

NOTE: Ignore Exercises 8, 9, 10, 11, 12 in section 4 of the workbook.

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a,b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx}+P(x)y=f(x)y^n$$

is called a **Bernoulli** equation.

The solution to a Bernoulli equation is obtained by setting $u=y^{1-n}$. Then u solves the first order linear ODE

$$\frac{du}{dx}+(1-n)P(x)u=(1-n)f(x),$$

and

$$y=u^{\frac{1}{1-n}}.$$

Example (from last class)

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

We set $u = y^{1-3} = y^{-2}$. This resulted in the equation for u

$$\frac{du}{dx} + 2u = 2e^{4x}$$

which has solutions

$$u = \frac{1}{2}e^{2x} + Ce^{-2x}.$$

Since $y = u^{-1/2}$, this gave

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}.$$

After applying the initial condition, we find $C = 1/2$ so that the solution to the IVP

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}} \quad \Longrightarrow \quad y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}.$$

Example

We wish to solve the ODE $\dfrac{dy}{dx} + \dfrac{2}{x}y = 8\sqrt{y}$.

$y^{1/2}$

- What is the value of n?

$n = \frac{1}{2}$

Example

We wish to solve the ODE $\dfrac{dy}{dx} + \dfrac{2}{x}y = 8\sqrt{y}$.

- What is the value of n?
- What is the new variable u?

$$u = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$$

Example

We wish to solve the ODE $\dfrac{dy}{dx}+\dfrac{2}{x}y=8\sqrt{y}$.

- What is the value of n?
- What is the new variable u?
- What is the first order ODE that u solves?

$$\frac{du}{dx} + ? = ?$$

$$\frac{du}{dx} + ?\,u = 4$$

Example

We wish to solve the ODE $\dfrac{dy}{dx} + \dfrac{2}{x}y = 8\sqrt{y}$.

- What is the value of n?
- What is the new variable u?
- What is the first order ODE that u solves?
- What is y in terms of u?

$$u = y^{\frac{1}{2}} \quad \Longrightarrow \quad y = u^2$$

Section 5: First Order Equations Models and Applications

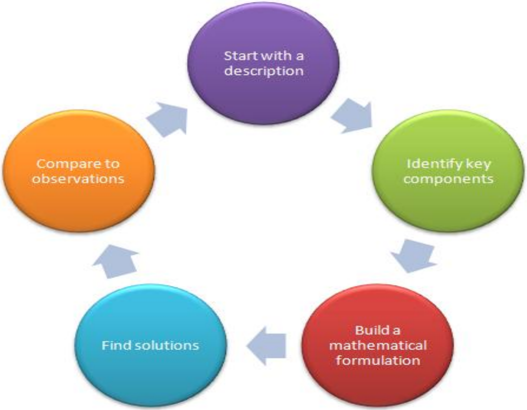

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let's call the population at time t $R(t)$.

R can be thought of as population density.

The rate of change of the population

$$\frac{dR}{dt} \propto R$$

rate is proportional to population

This gives an ODE $\frac{dR}{dt} = kR$ where k is some constant of proportionality.

If we take t in years with $t=0$ in 2011 then $R(0)=58$ and $R(1)=89$.

Together $\frac{dR}{dt} = kR$, $R(0)=58$ is an IVP. Separating the variables

$$\frac{1}{R}\frac{dR}{dt} = k$$

$$\int \frac{1}{R}\,dR = \int k\,dt$$

$$\ln|R| = kt + C$$

$$e^{\ln|R|} = e^{kt+C} = e^{c} e^{kt}$$

$$|R| = e^{c} e^{kt}, \quad \text{Let } A = \pm e^{C}$$

$$R = A e^{kt} \quad \text{a 1-parameter family}$$

Using $R(0) = 58$, $\quad 58 = A e^{0} = A$

The population $R(t) = 58 e^{kt}$.

we can find k from $R(1) = 89$

$$89 = 58 e^{k} \Rightarrow k = \ln\left(\frac{89}{58}\right)$$

Hence $R(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$

2021 corresponds to $t = 10$.

The expected population in 2021 is

$$R(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)}$$

$$\approx 4198$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

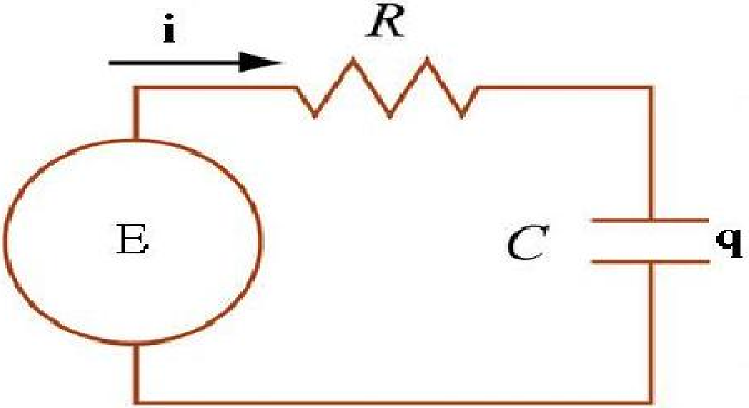

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

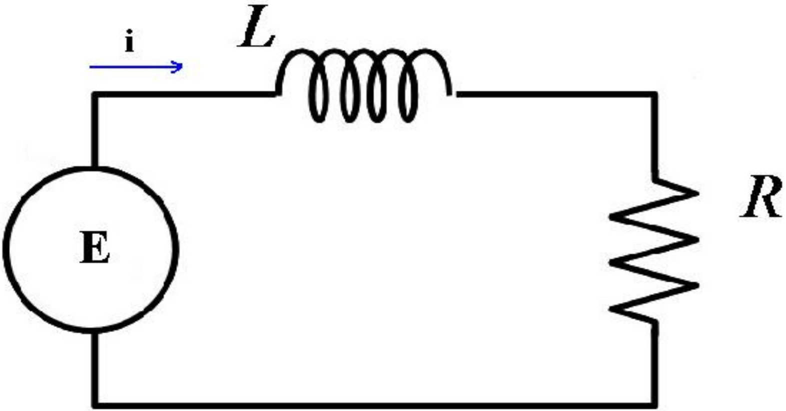

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Implied voltage E in volts (V),
Inductance L in henries (h), Charge q in coulombs (C),
Capacitance C in farads (f), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC - circuit

resistor capacitor Implied Voltage

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

1st order linear ODE $Rq' + \frac{1}{C}q = E$

LR - circuit

Inductor	Resistor	Implied Voltage
$L\frac{di}{dt}$ +	Ri =	$E(t)$

1st order linear ODE for i

$$Li' + Ri = E$$

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$R\frac{dq}{dt} + \frac{1}{C}q = E$, Here $E(t) = 200$

$R = 1000\,\Omega$

$C = 5 \cdot 10^{-6}$ f

$1000\frac{dq}{dt} + \frac{1}{5\cdot 10^{-6}}q = 200 \qquad q'(0) = 0.4$

Note $\frac{1}{5\cdot 10^{-6}} = \frac{10^6}{5} \Rightarrow \frac{10^6}{5(1000)} = \frac{10^3}{5} = 200$

In standard form: $\frac{dq}{dt} + 200q = \frac{1}{5}$

Here, $p(t) = 200$ so $\mu = e^{\int p(t)\,dt} = e^{200t}$.

$$e^{200t}\left(q' + 200q\right) = \frac{1}{5} e^{200t}.$$

$$\frac{d}{dt}\left[e^{200t} q\right] = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t}\, dt$$

$$e^{200t} q = \frac{1}{1000} e^{200t} + k$$

$$q = \frac{1}{1000} + k e^{-200t}$$

We can apply $q'(0) = \frac{2}{5}$

$$q'(t) = -200\,k\,e^{-200t}$$

$$q'(0) = -200\,k\,e^{0} = -200k = \frac{2}{5}$$

$$k = \frac{-1}{500}$$

The charge for $t>0$ is

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}.}$$

Looking at the long time behavior

$$\lim_{t\to\infty} q(t) = \frac{1}{1000}$$